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THE WASHING OF SOCKS

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1. The saving of water problem. A desert sheik knows from long experience that his socks must be washed in at least β cc of water in order to be comfortably clean. That is, the socks are sloshed around in β cc of water, wrung out, and dried. Ali, his wash boy, wishes to sell some of the water allowed him. How much water can he save by washing in steps and still return the socks sufficiently clean?

To answer this question we assume that the sloshing dissolves all the dirt completely and mixes it uniformly throughout the water being used at that step. It turns out that the algebra is simpler if the socks are assumed damp when given to Ali, and we make this assumption, too.

Suppose the damp socks hold α cc of water, and initially the dirty socks contain d gms of dirt. The sheik's instructions imply that "clean" socks will contain no more than

$$d^* = d\,\frac{\alpha}{\alpha+\beta}$$

grams of dirt.

Ali washes and wrings the socks successively in $\beta_1, \beta_2, \cdots, \beta_n$ cc of water. Let d_i be the number of grams of dirt in the socks after the ith washing. These numbers satisfy the difference equation

$$d_i = d_{i-1}\,\frac{\alpha}{\alpha+\beta_i}$$

with $d_0 = d$. The solution is evidently

$$d_n = d\,\frac{\alpha}{\alpha+\beta_1}\cdot\frac{\alpha}{\alpha+\beta_2}\cdots\frac{\alpha}{\alpha+\beta_n}\,. \tag{1}$$

To meet the sheik's requirements without waste $d_n = d^*$, implying

$$(\alpha+\beta_1)(\alpha+\beta_2)\cdots(\alpha+\beta_n) = \alpha^{n-1}(\alpha+\beta). \tag{2}$$

The amount of water used,

$$\beta_1+\beta_2+\cdots+\beta_n \tag{3}$$

can be minimized, subject to (2) by using traditional methods, such as Lagrange Multipliers, to find the minimum of a function of n variables subject to a constraint.

However, Ali notes that taking the nth root of both members of (2) shows that the geometric mean of the n numbers $\alpha+\beta_i$ is independent of β_i. Further, the amount of water used will be a minimum if the sum of the n numbers $\alpha+\beta_i$, and hence their mean, is a minimum.

Recall that the arithmetic mean of n numbers is never less than their geometric mean. Equality holds only if the n numbers are equal. *Hence* (3) *will be a minimum if and only if all the* β_i *are equal.*

Setting β_i equal in (2) and (3) Ali finds that *the amount*, S_n, *of water saved and available for sale, is n times the difference between the arithmetic and geometric means of the n numbers* $\alpha+\beta, \alpha, \alpha, \cdots, \alpha$. For

$$\begin{aligned} S_n &= \beta - \sum \beta_i \\ &= \beta - n\left\{(\alpha^{n-1}[\alpha+\beta])^{1/n} - \alpha\right\} \\ &= n\left\{\frac{(\alpha+\beta)+(n-1)\alpha}{n} - \left[\alpha^{n-1}(\alpha+\beta)\right]^{1/n}\right\}. \end{aligned}$$

Ali can also write

$$S_n = \beta - n\alpha\left\{[1+\beta/\alpha]^{1/n} - 1\right\}.$$

For n large, this involves an indeterminant form. Expanding the exponential Taylor series, he obtains

$$\begin{aligned} [1+\beta/\alpha]^{1/n} &= \exp \frac{1}{n} \ln (1+\beta/\alpha) \\ &= 1 + \frac{1}{n} \ln (1+\beta/\alpha) + O\left(\frac{1}{n^2}\right). \end{aligned}$$

Hence

$$S_n = \beta - \alpha \ln \left(1+\frac{\beta}{\alpha}\right) + O\left(\frac{1}{n}\right),$$

where $O(1/n)$ is negative and of order $1/n$. Hence the least upper bound for the relative amount saved is

(4) $$\lim_{n\to\infty} \frac{S_n}{\beta} = 1 - \frac{\alpha}{\beta} \ln \left(1+\frac{\beta}{\alpha}\right).$$

2. The maximum cleanliness problem. The sheik discovers what is going on. Furious, he tells Ali to use all the water to get the socks as clean as possible in n steps.

From his previous argument Ali knows that maximum cleanliness, i.e., minimum dirt, will be obtained by separating the available β cc into n equal parts, $\beta_i = \beta/n$.

By careful measuring Ali is surprised to find that the amount of water, α, to dampen the socks is exactly one cc. He replaces α with one in future formulas, and tells the sheik the formulas will apply to other socks on replacing β with β/α. Substituting in (1) shows that *using all the water, the minimum amount of dirt is*

(5) $$d_n' = d\left(1+\frac{\beta}{n}\right)^{-n}.$$

Ali notices that the best procedure for washing n times must get the socks at least as clean as the best procedure for washing $(n-1)$ times. (He argues that one possible n-wash routine is to use β cc in the best way for $n-1$ washes, and use no water in the nth wash.) Hence from (5), *$(1+\beta/n)^n$ and particularly $(1+1/n)^n$ is an increasing function of n.*

Considering the limit of (5) as $n\to\infty$ shows *the greatest lower bound for the amount of dirt which can be approached by multiple washings is*

$$d' = \lim_{n\to\infty} d_n' = de^{-\beta}. \tag{6}$$

3. Dynamic programming. Let $\delta(d, \beta, n)$ be the amount of dirt remaining of an original d grams if β cc of water are used in n washes after the initial wetting, and if the most efficient pattern of separating the water into n parts is followed. We may consider the water separated into two parts, with s cc to be used in t washings and then $\beta-s$ cc to be used in $n-t$ washings. Then δ must satisfy the difference equation

$$\delta(d, \beta, n) = \underset{s}{\text{minimum}}\, \delta[\delta(d, s, t), \beta - s, n - t] \tag{7}$$

and the function $\delta(d, \beta) = \lim_{n\to\infty} \delta(d, \beta, n)$ must satisfy

$$\delta(d, \beta) = \min_s \delta[\delta(d, s), \beta - s]. \tag{8}$$

It is easy to see that *$d(1+\beta/n)^{-n}$ satisfies* (7) *and $de^{-\beta}$ satisfies* (8). For, substituting $d(1+\beta/n)^{-n}$ for $\delta(d, \beta, n)$ in the right hand member of (7) yields

$$\min_s d\left(1 + \frac{s}{t}\right)^{-t}\left(1 + \frac{\beta - s}{n - t}\right)^{-n+t} \tag{9}$$

The derivative test shows that a minimum occurs for

$$s = t\frac{\beta}{n}\cdot$$

For this value of s, (9) reduces to

$$d\left(1 + \frac{\beta}{n}\right)^{-n}$$

completing the proof of the first part. The substitution of $de^{-\beta}$ in (8) is left as an exercise.

4. The *m* pair problem. The sheik, now on an economy drive, asks what happens if more than one pair of socks is washed. Ali finds the following. *Let m pairs of socks be washed one after the other in n equal portions of β/n cc. Let ${}_sd_i$ stand for the amount of dirt in pair s after i washings, $s=1, \cdots, m$; $i=1, \cdots, n$. Then ${}_sd_n$ and the limiting values ${}_sd_\infty$ are given by* (13) *and* (14) *below.*

Proof. If ${}_sW_i$ is the amount of dirt in the water while pair s is given washing i, then setting $(1+\beta/n)^{-1}=f$, the following are satisfied:

(10) $$ {}_sd_i = {}_sW_if $$

(11) $$ {}_sW_i = {}_sd_{i-1} + {}_{s-1}W_i(1-f) $$

and the partial difference equation

(12) $$ {}_sd_i = {}_sd_{i-1}f + {}_{s-1}d_i(1-f). $$

Equation (10) says the amount of dirt in the socks is the f proportion of that in the water. Equation (11) says the amount of dirt in the water is that brought by the socks plus that left in the water from the last step. Equation (12) follows by substitution. We define ${}_0W_i = {}_0d_i = 0$.

By direct substitution, the solution to (12) is seen to be

(13) $$ {}_sd_n = \Big\{ {}_sd_0 + {}_{s-1}d_0 n(1-f) + {}_{s-2}d_0 \frac{1}{2!} n(n+1)(1-f)^2 + {}_{s-3}d_0 \frac{1}{3!} n(n+1)(n+2)(1-f)^3 + \cdots + {}_1d_0 \frac{1}{(s-1)!} n(n+1) \cdots (n+s-2)(1-f)^{s-1} \Big\} f^n. $$

The limiting value as $n \to \infty$ is

(14) $$ {}_sd_\infty = \Big\{ {}_sd_0 + {}_{s-1}d_0\beta + {}_{s-2}d_0 \frac{1}{2!}\beta^2 + {}_{s-3}d_0 \frac{1}{3!}\beta^3 + \cdots + {}_1d_0 \frac{1}{(s-1)!}\beta^{s-1} \Big\} e^{-\beta}. $$

This concludes the proof.

The conclusion has the following interpretation. Since

$$ {}_1d_0 = {}_1d_0 \left[1 + \beta + \frac{1}{2!}\beta^2 + \frac{1}{3!}\beta^3 + \cdots \right] e^{-\beta}, $$

we may think of the dirt as separated into little packages proportional to

$$ e^{-\beta}, \beta e^{-\beta}, \frac{1}{2!}\beta^2 e^{-\beta}, \frac{1}{3!}\beta^3 e^{-\beta}, \cdots. $$

The effect of the washing is to leave the first package in the original pair of socks, put the next package in the second pair of socks, the third package in the third pair, and so on. The same is true for the effect of the washing on the dirt in any pair. Packages of dirt not placed in a later pair of socks are left in the water after the last washing.

In the end, the dirt remaining in any pair of socks is the first package of its original dirt, the second package from the pair just before it, the third package from the second pair ahead of it, and so on.

5. Minimum total amount of dirt. The sheik comments, "one of my socks is red, the other green. Instead of washing them both together in each wash, how much dirt will remain if at each step the red sock is washed alone, wrung out, and the green sock washed in this slightly dirty water?"

A modification of (14), replacing ${}_sd_0$ by $\frac{1}{2}d$ and β by 2β shows that r_∞ and g_∞, the amounts of dirt remaining in the red and green socks, are

$$r_\infty = \tfrac{1}{2}de^{-2\beta} \quad \text{and} \quad g_\infty = \tfrac{1}{2}d[1 + 2\beta]e^{-2\beta}.$$

The total amount of dirt remaining is

$$r_\infty + g_\infty = d[1 + \beta]e^{-2\beta} < de^{-\beta}.$$

While the amount of dirt in the red sock is always less than when the socks are washed together, the amount in the green sock will be less if and only if

$$(1 + 2\beta)e^{-2\beta} < e^{-\beta}.$$

This happens if $\beta > 1.25$, approximately.

Since the socks are not symmetrically treated, it might be that the total amount of dirt would be even less if the β cc were not divided evenly. Consider $n = 2$ first.

Suppose the available fluid is separated into two parts, β_1 and β_2, so $\beta_1 + \beta_2 = \beta$. Let $f_1 = (1 + 2\beta_1)^{-1}$, $f_2 = (1 + 2\beta_2)^{-1}$. Then if r_i, g_i are the amounts of dirt in the red and green socks after i washes,

$$r_i = r_{i-1}f_i, \qquad g_i = g_{i-1}f_i + r_i(1 - f_i).$$

After two washings, the total amount of dirt, using $r_0 = g_0 = d/2$, is

$$r_2 + g_2 = \frac{d}{2}[4 - f_1 - f_2]f_1f_2.$$

The condition $\beta_1 + \beta_2 = \beta$ becomes $1/f_1 + 1/f_2 = 2 + 2\beta = c$. Setting $f_1f_2 = z$, our problem is to minimize

$$r_2 + g_2 = \frac{d}{2}(4 - cz)z.$$

But the right member is a parabola opening downward with vertex at $z = 2/c = (1 + \beta)^{-1}$. *Hence any minimum comes at an end point.* A little algebra shows that z varies between $(1 + \beta)^{-2}$ and $(1 + 2\beta)^{-1}$. The end point of the range furthest from the vertex is at $z = (1 + \beta)^{-2}$. Hence the minimum value of the total amount of dirt will occur for this value of z, which corresponds to $\beta_1 = \beta_2$. This proves, for $n = 2$, *if two socks are washed, one after the other, in n changes of water, the minimum total amount of dirt remaining is reached by separating the available water equally among the n changes. This minimum is less than that obtained when the socks are washed together.*

A combination of induction and dynamic programming proves the above for $n > 2$. A separation of β cc of water into n portions, $\beta_1, \cdots, \beta_n$ generates a separation into two portions $\beta_1 + \beta_2 + \cdots + \beta_{n-2}$ and $\beta_{n-1} + \beta_n$. Similarly a separation into two portions is the first step in a separation into n on further separating the portions into $n-2$ and 2 parts. Now the separation into 2 portions generated by the most efficient separation into n is the most efficient separation into 2. For, if there were a more efficient separation into two portions, this would lead to a separation into n parts more efficient than the most efficient. The argument for $n=2$ shows that the most efficient separation of the second portion into two parts occurs when $\beta_{n-1}=\beta_n$.

Now we assume that the water is separated into two portions with the first to be separated into $n-1$ parts. The induction hypothesis says that the most efficient separation occurs when $\beta_1=\beta_2=\cdots=\beta_{n-1}$. Since we already had $\beta_{n-1}=\beta_n$, all n portions must be equal for minimum total dirt remaining.

6. The completely clean problem. The sheik asks if there isn't some way the socks can be washed completely clean. He knows from (6) that if the socks are completely dunked each time that the amount of dirt remaining is greater than $de^{-\beta}$. However, he suggests that only a portion of the sock be washed each time.

Suppose the socks be separated (mentally) into m equal portions, each portion holding $1/m$ cc when damp and initially containing d/m gms of dirt. Let the β cc of water be separated into n equal portions. By a magnificient feat of dexterity the m portions of the socks are washed one after the other in each of the n changes of water. A modification of (13) indicates that the amount of dirt left in part s is

$$ {}_sd_n^* = \frac{d}{m}\left\{1 + n(1-f^*) + \frac{1}{2!}n(n+1)(1-f^*)^2 + \cdots + \frac{1}{(s-1)!}n(n+1)\cdots(n+s-2)(1-f^*)^{s-1}\right\} f^{*n} $$

where $f^*=(1+m\beta/n)^{-1}$, $(1-f^*)=(m\beta/n)(1+m\beta/n)^{-1}$. The total dirt left is $\sum_{s=1}^{m} {}_sd^*_n$.

Naturally we seek the limit as $m, n\to\infty$. Since the terms in the series are all positive, by the theory of scrambled series [1], if the series converges for one arrangement it will converge to this value for all. We let $n\to\infty$ first, then $m\to\infty$. Using an adjusted version of (14), the total amount of dirt left is lower bounded by

$$ \frac{d}{m}\sum_{s=1}^{m}\left\{1 + m\beta + \frac{m^2\beta^2}{2!} + \cdots + \frac{(m\beta)^{s-1}}{(s-1)!}\right\} e^{-m\beta} \tag{15} $$

since

$$ \lim_{n\to\infty} f^{*n} = \lim\left(1+\frac{m\beta}{n}\right)^{-n} = e^{-m\beta} $$

$$ \lim_{n\to\infty} n(n+1)\cdots(n+s-2)\left(\frac{m\beta}{n}\right)^{s-1}\left(1+\frac{m\beta}{n}\right)^{-s+1} = (m\beta)^{s-1}. $$

(15) may be written as

$$
\begin{aligned}
(16)\qquad & d\left\{1 + (m-1)\beta + \frac{(m-2)m\beta^2}{2!} + \cdots + \frac{[m-(m-1)]m^{m-2}\beta^{m-1}}{(m-1)!}\right\} e^{-m\beta} \\
&= d\left\{(1-\beta)(1 + m\beta + \frac{(m\beta)^2}{2!} + \cdots + \frac{(m\beta)^{m-2}}{(m-2)!} + \frac{(m\beta)^{m-1}}{(m-1)!}\right\} e^{-m\beta} \\
&+ \beta \frac{(m\beta)^{m-1}}{(m-1)!} e^{-m\beta}.
\end{aligned}
$$

The series in parentheses may be regarded as the sum of terms from a Poisson distribution [2] with mean $\mu = m\beta$. It represents the probability that $0, 1, \cdots, (m-1)$ successes occur. For large μ the Poisson distribution is approximately normal with mean $\mu = m\beta$ and standard deviation $\sigma = \sqrt{m\beta}$. Hence the series

$$
(17)\qquad \left(1 + m\beta + \frac{(m\beta)^2}{2!} + \cdots + \frac{(m\beta)^{m-1}}{(m-1)!}\right) e^{-m\beta}
$$

is the probability that an event will occur whose standard score is less than

$$
t = \frac{m - m\beta}{\sqrt{m\beta}} = \sqrt{m}\left(\frac{1-\beta}{\sqrt{\beta}}\right).
$$

For β in the ranges $0<\beta<1$ or $\beta=1$ or $1<\beta$ the probabilities, and hence the values of (17) tend to 1 or 1/2 or 0 respectively as $m \to \infty$. The remaining term in (16) tends to zero. Hence, substituting in (16) we have *let a pair of socks with d grams of dirt be separated mentally into m parts, and the available β cc of water be separated into n parts. Let each of the m parts of the socks be washed successively in each of the n washes. Then the lower limit of the amount of dirt remaining is zero or $d(1-\beta)$ accordingly as $\beta \geqq 1$ or $\beta < 1$.*

This surprising result suggests that if we have only as much more water as the one cc necessary to dampen the socks in the first place, we can get the socks as clean as we please. Further, if we have less than 1 cc, we can remove up to that fraction of the dirt indicated by β.

Physically, this means that Ali should place one drop of water at the top of the damp sock, then work it all the way through the socks until it drips out the bottom. He repeats this until all β cc are used. In this way he will get the socks almost perfectly clean.

That $\beta = 1$ is large enough may be made intuitive. As the little drop moves along the socks it gradually becomes saturated to the same concentration as the water in the sock. Hence it drops off holding dirt at the initial concentration. If we persuade $\beta = 1$ cc to drop off the socks in little drops at the initial concentration, they will carry off all the dirt.

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A CURVE OF CONSTANT DIAMETER

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If C is a closed convex curve and P, P' are two distinct points on C such that the tangents at P and P' are parallel, then the segment PP' is called a diameter of C. If this diameter is of constant length for all such pairs P, P', then C is called a curve of constant diameter. The circle, of course, has this property, but there are also other curves of constant diameter. In fact, any involute of a closed differentiable curve with n cusps having one tangent in every direction is a curve of constant diameter. Furthermore, if this diameter be denoted by d, then the perimeter is $S=\pi d$ [1].

As an example let us consider the hypocycloid of three cusps, given by the parametric equations

$$(1) \qquad \begin{aligned} x &= 2\cos\theta + \cos 2\theta \\ y &= 2\sin\theta - \sin 2\theta \end{aligned} \qquad 0 \leqq \theta < 2\pi.$$

It is easy to see that this differentiable curve has exactly one tangent in every direction. We shall obtain a curve C of constant diameter as an involute of this curve.

The parametric equations of an involute of any curve are given by [2]

$$(2) \qquad \begin{aligned} x_1 &= x + \frac{x'}{\sqrt{\{(x')^2+(y')^2\}}} \int_\theta^{\theta_0} \sqrt{\{(x')^2+(y')^2\}}\, d\theta \\ y_1 &= y + \frac{y'}{\sqrt{\{(x')^2+(y')^2\}}} \int_\theta^{\theta_0} \sqrt{\{(x')^2+(y')^2\}}\, d\theta. \end{aligned}$$

(Here the primes denote differentiation with respect to the parameter θ.) Now we have

$$(3) \qquad x' = -2(\sin\theta + \sin 2\theta) = -4\sin\tfrac{3}{2}\theta\cos\tfrac{1}{2}\theta,$$

and

$$(4) \qquad y' = 2(\cos\theta - \cos 2\theta) = 4\sin\tfrac{3}{2}\theta\sin\tfrac{1}{2}\theta.$$

Hence

$$(5) \qquad \sqrt{\{(x')^2+(y')^2\}} = 4\sin\tfrac{3}{2}\theta,$$

and

$$(6) \qquad \int_\theta^{2/3\pi} \sqrt{\{(x')^2+(y')^2\}}\, d\theta = 8/3(1+\cos\tfrac{3}{2}\theta).$$

Substituting (3), (4), (5), and (6) in (2) we get $x_1=(2\cos\theta+\cos 2\theta)-8/3\cos\theta/2(1+\cos\frac{3}{2}\theta)=1/3(6\cos\theta+3\cos 2\theta-8\cos\theta/2-8\cos\theta/2\cos\frac{3}{2}\theta)=1/3(6\cos\theta+3\cos 2\theta-4(\cos 2\theta+\cos\theta)-8\cos\theta/2)=1/3(2\cos\theta-\cos 2\theta-8\cos\theta/2)$.

Also $y_1=2\sin\theta-\sin 2\theta+8/3\sin\theta/2(1+\cos\frac{3}{2}\theta)=1/3(6\sin\theta-3\sin 2\theta+8\sin\theta/2+8\sin\theta/2\cos\frac{3}{2}\theta)=1/3(6\sin\theta-3\sin 2\theta+4(\sin 2\theta-\sin\theta)+8\sin\theta/2)=1/3(2\sin\theta+\sin 2\theta+8\sin\theta/2)$.

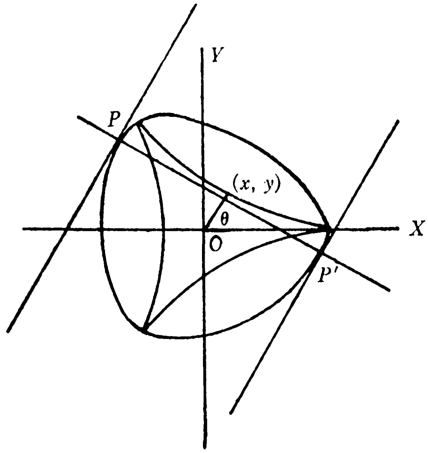

FIG. 1.

Hence the involute of (1) is the curve C given by the parametric equations

$$(7)\qquad \begin{aligned} x_1 &= 1/3(2\cos\theta - \cos 2\theta - 8\cos\theta/2) \\ y_1 &= 1/3(2\sin\theta + \sin 2\theta + 8\sin\theta/2). \end{aligned} \qquad 0 \leqq \theta < 4\pi.$$

We will now verify that the curve (7) has a constant diameter and that the perimeter of this curve is π times this diameter: Let $P(x_1(\theta), y_1(\theta))$ be the coordinates of an arbitrary point on the curve C. Then $P'(x_1(\theta+2\pi), y_1(\theta+2\pi))$ are the coordinates of the corresponding opposite point. Thus the length d of the diameter is given by

$$d = \sqrt{\{(x_1(\theta + 2\pi) - x_1(\theta))^2 + (y_1(\theta + 2\pi) - y_1(\theta))^2\}} = 16/3(\cos^2\theta/2 + \sin^2\theta/2) = 16/3.$$

The perimeter S is given by

$$(8)\qquad S = \int_0^{4\pi} \sqrt{\{(x_1')^2 + (y_1')^2\}}\, d\theta.$$

But $x_1' = 4/3 \sin\theta/2(1+\cos\frac{3}{2}\theta)$ and $y_1' = 4/3\cos\theta/2(1+\cos\frac{3}{2}\theta)$. Therefore

$$\sqrt{\{(x_1')^2 + (y_1')^2\}} = 4/3(1 + \cos\tfrac{3}{2}\theta).$$

Hence

$$S = 4/3\int_0^{4\pi}(1 + \cos\tfrac{3}{2}\theta)\, d\theta = 16/3\pi.$$

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A THEOREM OF SCHEMMEL

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1. Schemmel [3] generalized the Euler ϕ-function in the following way. Let k be a fixed integer $\geqq 1$ and let $\Phi_k(n)$ denote the number sets of k consecutive integers each less than n and relatively prime to n. Then we have

$$\Phi_k(n) = n \prod_{p|n} \left(1 - \frac{k}{p}\right),$$

provided each prime divisor of n is greater than k; otherwise it is evident that $\Phi_k(n) = 0$. For proof of this and related results see for example Bachmann [**1**, pp. 91–94]. For additional references see Dickson [**2**, p. 147].

Schemmel also stated the following result. Let $1 \leqq r \leqq k$ and let $P = P_{n,k,r}$ denote the product of the r-th terms of the $\Phi_k(n)$ sets of consecutive integers described above. If $k=1$, then $P \equiv \pm 1 \pmod{n}$ by Wilson's theorem. If $n > 1$ we have

$$P^{k-1} \equiv \{(-1)^{r-1}(r-1)!(k-r)!\}^{\Phi_k(n)} \pmod{n}. \tag{1}$$

In particular when $r=1$, $k=2$, (1) becomes

$$P \equiv 1 \pmod{n}. \tag{2}$$

In the present note we prove the following results.

THEOREM 1. *Let n be odd. Then*

$$\prod a \equiv 1, \qquad \prod (a+1) \equiv -1 \pmod{n}, \tag{3}$$

where in each case the product is extended over all a such that $1 \leqq a < n$ and $(a(a+1), n) = 1$.

THEOREM 2. *Let $(n, 6) = 1$. Then*

$$\prod (a+1) \equiv 1 \pmod{n}, \tag{4}$$

where the product is extended over all a such that $1 \leqq a < n-1$ and $(a(a+1)(a+2), n) = 1$.

The second theorem is presumably new.

2. *Proof of Theorem* 1. Let $1 \leqq a < n$ and $(a(a+1), n) = 1$, where $(n, 2) = 1$, $n > 1$. Then there exist integers b, c such that

$$ab \equiv (a+1)c \equiv 1 \pmod{n} \qquad (1 \leqq b < n,\ 1 \leqq c < n). \tag{5}$$

Then $(b+1)a \equiv a+1 \pmod{n}$, so that both b and $b+1$ are prime to n. Moreover if $a \equiv b$ we have $a \equiv b(a+1)c \equiv (a+1)c \equiv 1 \pmod{n}$. It follows that

$$\prod a = \prod_{a \neq 1} a \equiv 1 \pmod{n}.$$

Next, it follows from (5) that $(c-1)(a+1) \equiv c(a+1) - (a+1) \equiv -a \pmod{n}$, so that both $c-1$ and c are prime to n. Also if $a+1 \equiv c$ we have $a+1 \equiv cab \equiv (1-c)b \equiv -ab \equiv -1 \pmod{n}$. Therefore

$$\prod (a+1) = (n-1) \prod_{a+1 \neq n-1} (a+1) \equiv -1 \pmod{n}.$$

This evidently completes the proof of Theorem 1.

3. *Proof of Theorem* 2. Let $1 \leqq a < n-1$ and $(a(a+1)(a+2), n) = 1$, where $(n, 6) = 1$, $n > 1$. Then there exist integers b, c, d such that

$$ab \equiv (a+1)c \equiv (a+2)d \equiv 1 \pmod{n} \qquad (1 \leqq b, c, d < n). \tag{6}$$

It follows from (6) that $(c+1)(a+1) \equiv a+2$, $(c-1)(a+1) \equiv -a \pmod{n}$, so that $c-1$, c, $c+1$ are all prime to n. Moreover if $a+1 \equiv c$ we have $a+1 \equiv cab \equiv (1-c)b \equiv -ab \equiv -1$, so that $a+2 \equiv 0$ which contradicts the hypothesis. Therefore

$$\prod (a+1) \equiv 1 \pmod{n}.$$

This evidently proves Theorem 2.

Supported in part by NSF grant GP-1593.

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1. P. Bachmann, Niedere Zahlentheorie, vol. 1, Leipzig, 1902.
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ON SQUARES IN ARITHMETIC PROGRESSION

MERTON TAYLOR GOODRICH, Keene, New Hampshire

The number triplet (1, 25, 49) has entries that are the squares of integers and are also consecutive elements of an arithmetic progression. This is also true for the triplet (289, 625, 961). It is the purpose of this paper to establish methods of generating triplets which have these properties.

Given a triplet (M_1, M_2, M_3), with $M_1 < M_2 < M_3$, which satisfies the above mentioned criteria, it follows that $M_1 = x^2$, $M_2 = (x+a)^2$, and $M_3 = (x+ar/s)^2$, $s > 0$, where the common difference is $d = 2ax + a^2$. Now since $M_3 - M_1 = 2d$, it follows that $2(2ax+a^2) = (2axrs + a^2r^2)/s^2$ or, $x = a(r^2 - 2s^2)/2s(2s-r)$. Substitute this value of x into d, and obtain $d = a^2r(r-s)/s(2s-r)$.

If a, r, and x are greater than zero, it follows that $s\sqrt{2} < r < 2s$. Further, if a and x are greater than zero and $r < 0$, it follows that $-r > s\sqrt{2}$.

Now if M_1, M_2, and M_3 are in arithmetic progression, then

$$N_i = 4s^2(2s-r)^2 M_i/a^2, \qquad i = 1, 2, 3,$$

would be in arithmetic progression with common difference $D = 4s^2(2s-r)^2 d/a^2$.

This gives

$$N_1 = (r^2 - 2s^2)^2 \tag{1}$$

$$D = 4sr(r-s)(2s-r). \tag{2}$$

$$\prod (a+1) = (n-1) \prod_{a+1 \neq n-1} (a+1) \equiv -1 \pmod{n}.$$

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Now if M_1, M_2, and M_3 are in arithmetic progression, then

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would be in arithmetic progression with common difference $D = 4s^2(2s-r)^2d/a^2$.

This gives

(1) $$N_1 = (r^2 - 2s^2)^2$$

(2) $$D = 4sr(r-s)(2s-r).$$

If x, a, and r are larger than zero and $s\sqrt{2}<r<2s$, as noted earlier, then these formulas give a two parameter family of triplets of the desired kind. Triplets of this type will be said to belong to Group One.

If x and a are positive, $r<0$, and $-r>s\sqrt{2}$, then these formulas give a two parameter family of triplets of the desired kind. Triplets of this type will be said to belong to Group Two.

For each pair of values of s and r, chosen with the above restrictions, we may always find a set of squares in arithmetic progression.

For Group Two, if we substitute $-r$ for r in (2) we obtain, $D=4sr(r+s)(2s+r)$.

All the squares in a set may be multiplied by the same square.

If $r=2f$ and $f<s$, the squares in the set may each be divided by 4.

If g is a divisor of both r and s, the squares in the set are multiples of g^4.

As a check on the results, or to find the ratio r/s when the squares are given, we may use the relationship $r/s=(\sqrt{N_3}-\sqrt{N_1})/(\sqrt{N_2}-\sqrt{N_1})$ with $\sqrt{N_3}>0$, in Group One and $\sqrt{N_3}<0$, in Group Two.

EXAMPLES

Group One $r>0$

s	r	D	N_1	N_2	N_3	roots
2	3	24	1	25	49	1, 5, 7
3	5	120	49	169	289	7, 13, 17
4	7	336	289	625	961	17, 25, 31
5	8	960	196	1156	2116	14, 34, 46
7	10	3360	4	3364	6724	2, 58, 82
7	11	3696	529	4225	7921	23, 65, 89
7	12	4×840	4×529	4×1369	4×2209	46, 74, 94
9	15	81×120	81×49	81×169	81×289	63, 117, 153

Group Two $r<0$

s	r	D	N_1	N_2	N_3	roots
1	−3	240	49	289	529	7, 17, −23
1	−5	840	529	1369	2209	23, 37, −47
2	−3	840	1	841	1681	1, 29, −41
2	−5	2520	289	2809	5329	17, 53, −73
4	−7	18480	289	18769	37249	17, 137, −193
7	−10	114240	4	114244	228484	2, 338, −478
9	−15	81×5280	81×49	81×5329	81×10609	63, 657, −927

Checks by finding r/s from roots

(7, 11) $r/s=(89-23)/(65-23)=66/42=11/7$

(9, 15) $r/s=(153-63)/(117-63)=90/54=15/9$

(9, −15) $r/s=(-927-63)/(657-63)=-990/594=-15/9$.

The author wishes to express his gratitude and appreciation to the referee for his valuable suggestions for improving this paper.

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ANOTHER LOOK AT DIFFERENTIATION

JEROME H. MANHEIM, Northern Michigan University

Characteristically the definition of the derivative, dy/dx, follows closely upon the discussion of the limit of a function. And just as characteristically, most of the students who really understand why

$$\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)=6, \tag{1}$$

fail in their understanding of the "delta" process. As an aid to understanding the former situation, we sketch a graph so that we can "see" how the function assumes a sequence of values with 6 as a limit, if the independent variable takes on a sequence of values having 3 as a limit. Thus the limit value is exactly the value the function would have were it continuous at $x=3$, i.e., were the hole "filled in." Conventionally, we fail to make the corresponding explanation for the derivative. The purpose here is to exploit the analogy between the "limit" that appears in (1) and the "limit" that appears in

$$\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right].$$

Towards this end we analyze, first, the "delta" process for the function given by $f(x)=x^2$.

(1) $y=f(x)=x^2$

(2) $y+\Delta y=f(x+\Delta x)=(x+\Delta x)^2$

(3) $\Delta y=f(x+\Delta x)-f(x)=2x\Delta x+\overline{\Delta x}^2$

(4) $\dfrac{\Delta y}{\Delta x}=\dfrac{f(x+\Delta x)-f(x)}{\Delta x}=(2x+\Delta x)\dfrac{\Delta x}{\Delta x}=2x+\Delta x,\quad \textit{if } \Delta x\neq 0$

(5) $\dfrac{dy}{dx}=\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}=\lim\limits_{\Delta x\to 0}(2x+\Delta x)=2x.$

The difficulty is, of course, in making the transition from step (4) to step (5), for the former requires that Δx not be zero while, in the latter, the same Δx is treated as if it were zero.

Now, in step (4) we have a function of two variables; that is, $\Delta y/\Delta x$ is a function of x and Δx. However, our focus is always on the derivative at x, and x remains fixed throughout the differentiation process, (though, of course, it is a perfectly general abscissa within the domain of definition). In other words, Δx is the only true variable in the differentiation process. Thus, we can view $\Delta y/\Delta x=(2x+\Delta x)\Delta x/\Delta x$ as a function of the single variable Δx. The graph of this function is shown in Fig. 1 for various values of x. Naturally, because of the factor $\Delta x/\Delta x$, there is a discontinuity ("hole" type) at $\Delta x=0$ in each case.

In Step (5) we are asked to evaluate $\lim_{\Delta x\to 0}\Delta y/\Delta x$. Relying upon our discussion for the limit of a function, we see that this limit can be found by con-

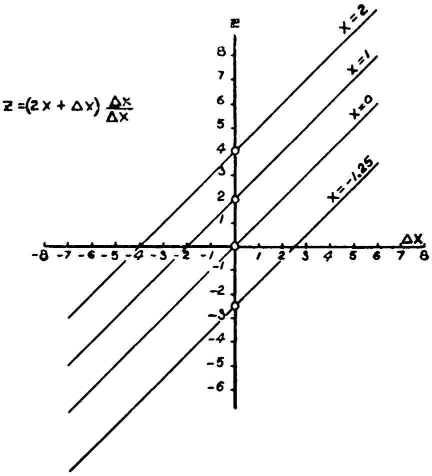


FIG. 1.

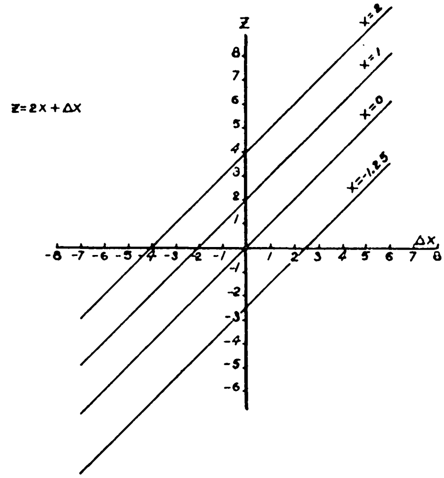


FIG. 2.

sidering the value, at $\Delta x=0$, of the set of functions identical with $\Delta y/\Delta x$ (one for each value of x) except at $\Delta x=0$. The new functions are to "fill in" at $\Delta x=0$; that is, they are continuous there. The latter functions are of the form, $2x+\Delta x$, there being one such function for each value of x. A few of these are viewed in Fig. 2. The student can now "see" that the derivative of x^2 with respect to x at $x=a$ is the ordinate of the "filled in" hole, that is, the value of the function $2a+\Delta x$ at $\Delta x=0$. For $f(x)=x^3$ the reader can readily verify that the figures for the difference quotients and corresponding continuous functions (graphs with ordinates $(3x^2+3x\Delta x+\overline{\Delta x}^2)\Delta x/\Delta x$ and $(3x^2+3x\Delta x+\overline{\Delta x}^2)$ respectively) are as shown in Figs. 3(a) and 3(b).

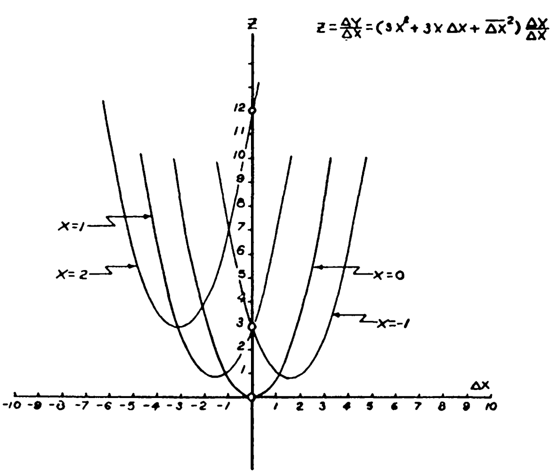


FIG. 3(a).

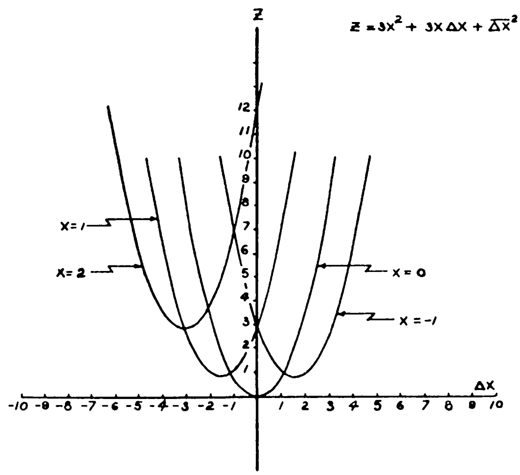


FIG. 3(b).

These graph pairs serve several purposes. In the first place, of course, they focus attention on the continuous functions that are related to the difference quotients. (With this in mind one might apply this approach to finding the derivative of $f(x)=x^{1/3}$ and VIEW the situation for $x=0$.) By establishing the correspondence between the "limits" in "limit of a function" and "limit of the difference quotient," the groundwork is laid for the transfer of understanding. The graphs facilitate the change from x to Δx as the variable in the two situations. Finally, this is "one more approach," and a notion conceptually as difficult as the derivative (and as important) cannot be overcooked for most students.

AN EXTENSION OF THE MEAN-VALUE THEOREM IN E_n

CHIH-CHIN LAN, National Taiwan University

THEOREM. *Assume that f, $g \in C'$ in the open set $S \subset E_n$. Let X and Y be two points of S such that the line segment $L(X, Y) \subset S$, where*

$$L(X, Y) = \{Z \mid Z = \theta X + (1-\theta)Y,\ 0 < \theta < 1\},$$

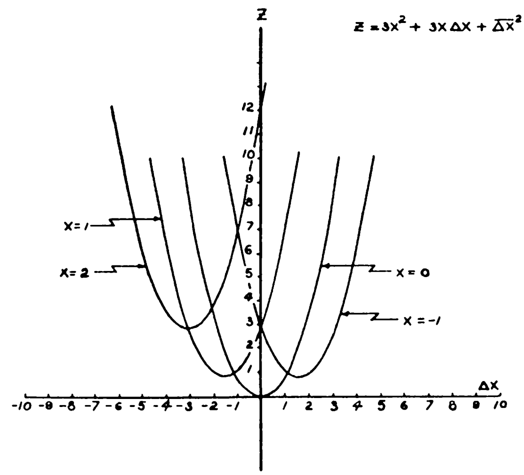
and u is the unit vector along $L(X, Y)$. Let f and g satisfy the following conditions:

(i) *At any point $Z_0 \in L(X, Y)$, we have either $D_u g(Z_0) \neq 0$ or $D_u f(Z_0) \neq 0$ where $D_u g$ and $D_u f$ are the directional derivatives.*

(ii) $g(X) \neq g(Y)$.

Then for any positive integer m, there exist m distinct points $Z_i \in L(X, Y)$ $(i=1, 2, \cdots, m)$, such that

$$\frac{f(Y)-f(X)}{g(Y)-g(X)} = \frac{1}{m}\sum_{i=1}^{m}\frac{\nabla f(Z_i)\cdot(Y-X)}{\nabla g(Z_i)\cdot(Y-X)}.$$



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(i) *At any point* $Z_0\in L(X, Y)$, *we have either* $D_ug(Z_0)\neq 0$ *or* $D_uf(Z_0)\neq 0$ *where* D_ug *and* D_uf *are the directional derivatives.*

(ii) $g(X)\neq g(Y)$.

Then for any positive integer m, *there exist* m *distinct points* $Z_i\in L(X, Y)$ $(i=1, 2, \cdots, m)$, *such that*

$$\frac{f(Y)-f(X)}{g(Y)-g(X)} = \frac{1}{m}\sum_{i=1}^{m}\frac{\nabla f(Z_i)\cdot(Y-X)}{\nabla g(Z_i)\cdot(Y-X)}.$$

Before we present a proof, we will first establish the following lemma which constitutes a special case of our theorem for $n=1$.

LEMMA. *Let f and g be two continuous functions defined on the closed interval $[a, b]$, and assume that each function has a derivative at each point of the open interval (a, b). Assume further that g and f satisfy the following conditions:*

(i) *At any point $X_0 \in (a, b)$, we have either $g'(X_0) \neq 0$ or $f'(X_0) \neq 0$.*

(ii) $g(a) \neq g(b)$.

Then, for any positive integer m, there exist m distinct points $z_i \in (a, b)$, such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{1}{m} \sum_{i=1}^{m} \frac{f'(z_i)}{g'(z_i)}.$$

Proof. We may assume that $g(b)-g(a)>0$. Let

$$r = \frac{g(b) - g(a)}{m} \text{ and } r_k = g(a) + \frac{k(g(b) - g(a))}{m} = g(a) + kr,\ k = 0, 1, 2, 3 \cdots m.$$

Then $r = r_k - r_{k-1} > 0$.

We have $g(a) = r_0$, $g(b) = r_m$. To unify the notation, let $a = x_0$, $b = x_m$. Since $g(x)$ is continuous and since $g(a) = r_0 < r_1 < \cdots < r_m = g(b)$, there exists an x_1 in $a < x < b$ such that $g(x_1) = r_1$, an x_2 in $x_1 < x < b$ such that $g(x_2) = r_2$, an x_3 in $x_2 < x < b$ such that $g(x_3) = r_3$, etc. Thus, we obtain a sequence $x_0 < x_1 < x_2 \cdots < x_m$ such that $g(x_i) = r_i$ for $i = 0, 1, 2, \cdots, m$.

Now consider the closed interval $[x_i, x_{i+1}]$, $i = 0, 1, 2, \cdots, m-1$. By Cauchy's mean-value theorem, there is at least one point $z_{i+1} \in (x_i, x_{i+1})$ such that

$$[f(x_{i+1}) - f(x_i)]g'(z_{i+1}) = [g(x_{i+1}) - g(x_i)]f'(z_{i+1}) = rf'(z_{i+1}).$$

By condition (i), we have either $f'(z_{i+1}) \neq 0$ or $g'(z_{i+1}) \neq 0$ at the point z_{i+1}. By condition (ii), we have $r \neq 0$. If $f'(z_{i+1}) \neq 0$, then $[f(x_{i+1}) - f(x_i)]g'(z_{i+1}) \neq 0$. This implies $g'(z_{i+1}) \neq 0$ in both cases. Hence

$$f(x_{i+1}) - f(x_i) = r \frac{f'(z_{i+1})}{g'(z_{i+1})}, \quad \text{for } i = 0, 1, \cdots m - 1.$$

By addition

$$f(b) - f(a) = r\left[\sum_{i=1}^{m} \frac{f'(z_i)}{g'(z_i)}\right] = \frac{g(b) - g(a)}{m}\left(\sum_{i=1}^{m} \frac{f'(z_i)}{g'(z_i)}\right).$$

Proof of the theorem.

Keep X and Y fixed, and define h and P as follows: $h(t) = f(X+tu)$, $P(t) = g(x+tu)$, $0 \leq t \leq \rho$, where $\rho = |Y - X|$ and u is the unit vector given by $u = Y - X/\rho$. Then, $X + \rho u = Y$ and we have $f(Y) - f(X) = h(\rho) - h(0)$, $g(y) - g(X) = P_{(\rho)} - P_{(0)}$. By differentiating $h(t)$ and $P(t)$, we get $h'(t) = D_u f(X+tu) = Vf(X + tu) \cdot u = Vf(Z) \cdot u$ and $P'(t) = Dg(X + tu) = Vg(X + u) \cdot u - Vg(Z) \cdot u$ where $Z = X + tu$.

It is obvious that $h(t)$ and $P(t)$ satisfy the conditions of the preceding lemma. Hence for any m there exist m distinct points $t_i \in (0, \rho)$ such that

$$h(\rho) - h(0) = [P(\rho) - P(0)]\left[\frac{1}{m}\sum_{i=1}^{m}\frac{h'(t_i)}{P'(t_i)}\right].$$

Thus

$$f(Y) - f(X) = [g(Y) - g(X)]\left[\frac{1}{m}\sum_{i=1}^{m}\frac{Vf(Z_i)\cdot u}{Vg(Z_i)\cdot u}\right]$$
$$= [g(Y) - g(X)]\left[\frac{1}{m}\sum_{i=1}^{m}\frac{Vf(Z_i)\cdot(Y - X)}{Vg(Z_i)\cdot(Y - X)}\right],$$

where $Z_i = X + t_i u \in L(X, Y)$. This proves the theorem.

For $g(X) = x_1 + x_2 + \cdots + x_n$ where $X = (x_1, x_2, \cdots, x_n)$, we obtain

$$Vg(Z)\cdot(Y - X) = \sum_{i=1}^{n}(y_i - x_i) = g(Y) - g(X),$$

and, consequently,

$$f(Y) - f(X) = \frac{1}{m}\sum_{i=1}^{m} Vf(Z_i)\cdot(Y - X).$$

If we take $m = 1$, then we obtain the so-called mean-value theorem.

SHORTCUT TO SUMMATION OF INFINITE SERIES

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Many problems in applied mathematics yield solutions in the form of infinite series. The practical usefulness of such solutions is somewhat limited by the amount of work involved in the computations; this is particularly true in the case of those investigators to whom no digital computer is available.

This paper deals with a specific category of infinite series to which a special summation procedure can be applied. The method is approximate but extremely simple.

Method of attack. It is proposed to compute the sum of an infinite series by adding several terms and an approximation to the remainder. No universality is claimed for the procedure developed in this paper; the limitations are self-evident and require no discussion.

The approximate expression for the remainder is derived by operating with a function $f(x)$ which has the following characteristics:

In the interval $1/2 \leqq x < \infty$, $f(x)$ is continuous and monotonic, and has a continuous derivative.

The improper integral $\int_{1/2}^{\infty} f(x)dx$ exists.

if one replaces the area BCD by the area of a triangle. Thus, one obtains the following approximation for the sum of the series:

$$s \cong f(\tfrac{1}{2}) - f(1) + f(1\tfrac{1}{2}) - f(2) + \cdots + f(k - \tfrac{1}{2}) - f(k) + R, \tag{3}$$

where R is given by (2).

Reference to the function $f(x)$ may now be dropped and the series can be written in the conventional form,

$$a_1 - a_2 + a_3 - a_4 + \cdots;$$

the remainder after n terms may be expressed as follows:

$$R_n = 3/4a_{n+1} - 1/4a_{n+2}. \tag{4}$$

Example 2. It is desired to compute the sum of the series

$$1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + 1/7 - 1/8 + \cdots.$$

One decides to use 6 terms and an approximation to the remainder. Making use of (4), one obtains:

$$s \cong 1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + 3/4(1/7) - 1/4(1/8) = 0.6926.$$

The precise value of the sum is $s = \ln 2 = 0.6932$. The accuracy of the method in this particular case is 0.1 percent. Using 6 terms without the remainder, one would have $s \cong 0.6167$; and using 8 terms, $s \cong 0.6345$.

Remarks. The formulas developed in this paper are special cases of Euler's transformations of infinite series. Those readers who desire a more analytical treatment are referred to F. B. Hildebrand's "Introduction to Numerical Analysis" (McGraw-Hill, 1956).

AN EXACT PERIMETER INEQUALITY FOR THE PEDAL TRIANGLE

A. ZIRAKZADEH, University of Colorado

Introduction. In 1775 in the *Acta Eruditorum*, J. F. de Fuschis a Fagnano proved the following theorem:

Among all the triangles inscribed in a given acute-angled triangle, the triangle that has for its vertices the feet of the altitudes of the given triangle has the least perimeter.

Fagnano proved this theorem using differential calculus. Other proofs, which were geometric in nature, appeared later, the most ingenious and elegant one being the proof given by H. A. Schwarz. Several proofs of Fagnano's theorem, including Schwarz's proof are given in [1].

In this paper we consider the following problem: Suppose the given acute-angled triangle ABC has perimeter l and the inscribed triangle which has the smallest perimeter has perimeter p. What are the maximum and minimum values of the ratio p/l? It is clear that the ratio p/l will approach zero as one of the angles of triangle ABC approaches zero, the two included sides being held constant in length. On the other hand, it can be easily shown that $p/l<\frac{2}{3}$. (For example, use proposition exercise 6, p. 99 of [2].) But not much more than this seems to be known about the problem.

In this paper we shall prove that $p/l\leqq\frac{1}{2}$ and that the equality holds if and only if ABC is an equilateral triangle.

The analytical proof of this theorem is hopelessly involved, and the author has not been able to find a purely geometric proof either. But a combination of geometric arguments and analytical technics leads to an elementary and simple proof of the theorem.

THEOREM. *Given an acute-angled triangle with perimeter l. If the inscribed triangle of smallest perimeter has perimeter p, then $p/l\leqq\frac{1}{2}$. The equality holds if and only if the given triangle is equilateral.*

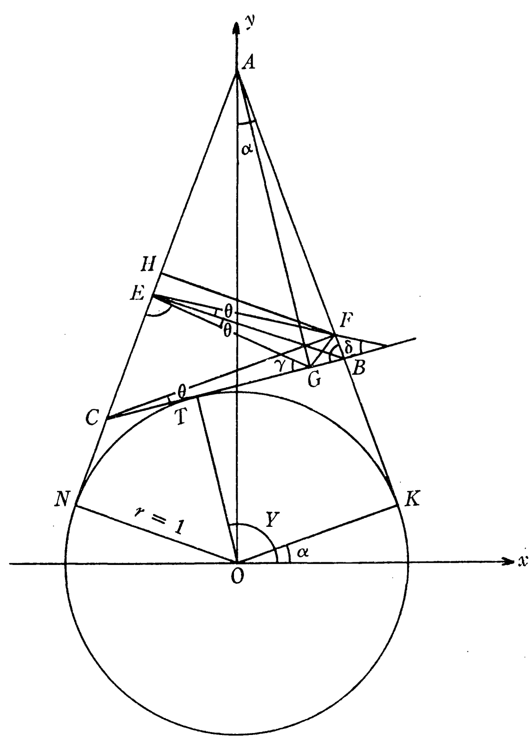

FIG. 1.

Proof. Consider the acute-angled triangle ABC (see Fig. 1); let $\angle BAC=2\alpha<\pi/2$, and draw the escribed circle with center O, opposite vertex A. Let side CB of triangle ABC rotate about O, CB being always tangent to the circle, and such that angles B and C remain less than $\pi/2$. For each position of BC there will be a corresponding inscribed triangle EFG of least perimeter. As BC changes the perimeter l of triangle ABC will remain the same while the perimeter p of

EFG will change. We shall prove that p attains its maximum when ABC is an isosceles triangle. Since l remains the same, it follows that p/l will also attain its maximum for that position of ABC.

Let OA be taken as the y-axis and OL, perpendicular to OA, as the x-axis of a cartesian coordinate system in the plane of triangle ABC, and let the radius of the escribed circle O be 1 unit. Draw OT perpendicular to CB and let ϕ represent the angle between the positive x-axis and OT. We shall evaluate the perimeter p of EFG in terms of α and ϕ and show that p attains its maximum when $\phi=\pi/2$.

Note that AG, CF, and BE are bisectors of $\angle EGF$, $\angle EFG$, and $\angle FEG$ respectively and let θ represent $\angle GEB$, γ represent $\angle CGE$, and δ represent the angle between EF and CB. It follows immediately that

$$EG + GF = \frac{EF\cos\delta}{\cos\gamma} \text{ and that } p = EF + EG + GF = EF\left(1 + \frac{\cos\delta}{\cos\gamma}\right).$$

Next we prove that $\gamma=2\alpha$ and $\delta=2\phi-\pi$.

Note that $\angle A+\angle B+\angle C=\pi$ and $\angle C+\angle E+\gamma=\pi$. But $\angle BEC=\angle E+\theta=\pi/2$ and $\angle B+\angle BCF=\angle B+\theta=\pi/2$, where $\angle E$ represents $\angle CEG$ and θ represents $\angle BEG=\angle BCF$. (B, C, E, and F lie on a circle). Therefore $\angle E=\angle B$ and hence $\angle A=2\alpha=\gamma$.

Next we notice that $\delta=\gamma-2\theta=2\alpha-2\theta$. So we must prove that $2\alpha-2\theta=2\phi-\pi$ or $\phi-\alpha=\pi/2-\theta$. But $\phi-\alpha=\angle B$ and $\angle B=\pi/2-\theta$. So $\phi-\alpha=\pi/2-\theta$ and $\delta=2\phi-\pi$. Thus we have proved that $p=EF(1-\cos 2\phi/\cos 2\alpha)$.

Next we shall evaluate EF in terms of ϕ and α.

Draw a line, through F and perpendicular to AC, to intersect AC in H. Then $EF=FH/\cos\theta$. But $FH=CF\sin\angle ECF=CF\cos 2\alpha$ and $CF=AC\sin 2\alpha$. Therefore

$$EF = AC\,\frac{\cos 2\alpha\sin 2\alpha}{\cos\theta} = AC\,\frac{\cos 2\alpha\sin 2\alpha}{\sin(\phi-\alpha)},$$

since we have proved that $\phi-\alpha=\pi/2-\theta$. Noting that $AC=AN-NC=[\cot\alpha-\cot\frac{1}{2}(\phi+\alpha)]$ we have

$$EF = \frac{\cos 2\alpha\sin 2\alpha}{\sin(\phi-\alpha)}\left[\cot\alpha - \cot\frac{1}{2}(\phi+\alpha)\right].$$

This can be shown, by trigonometric identities, to be

$$EF = \frac{2\cos 2\alpha\cos\alpha}{\sin\alpha+\sin\phi}.$$

Therefore

$$(1)\qquad p = EF\left(1-\frac{\cos 2\phi}{\cos 2\alpha}\right) = \frac{2\cos 2\alpha\cos\alpha}{\sin\alpha+\sin\phi}\left(1-\frac{\cos 2\phi}{\cos 2\alpha}\right).$$

Since it was assumed that ABC is acute-angled, $\cos\alpha$ and $\cos 2\alpha$ are always different from zero and, hence, p always exists.

Next we have to find those values of ϕ for which p attains its maximum. By evaluating $dp/d\phi$ and equating it to zero we obtain the following equation,

$$\frac{2\sin 2\phi}{\sin\alpha+\sin\phi}-\frac{\cos\phi(\cos 2\alpha-\cos 2\phi)}{(\sin\alpha+\sin\phi)^2}=0, \tag{2}$$

and since $\sin\alpha+\sin\phi\neq 0$, this leads to the equation

$$2\sin 2\phi-\frac{\cos\phi(\cos 2\alpha-\cos 2\phi)}{\sin\alpha+\sin\phi}=0 \tag{3}$$

or

$$\cos\phi(\sin^2\phi+2\sin\phi\sin\alpha+\sin^2\alpha)=\cos\phi(\sin\phi+\sin\alpha)^2=0. \tag{4}$$

Therefore p has a maximum or minimum value of

$$\frac{2\cos 2\alpha\cos\alpha}{\sin\alpha+1}\left(1+\frac{1}{\cos 2\alpha}\right)=\frac{4\cos^3\alpha}{1+\sin\alpha}\quad\text{at}\quad\phi=\frac{\pi}{2}$$

and has no other maxima or minima.

To prove that p attains its maximum at $\phi=\pi/2$, we notice that p is a continuous function of ϕ in the interval $\pi/2-\alpha\leqq\phi\leqq\pi/2+\alpha$. Furthermore, as $\phi\to\pi/2+\alpha$, vertices F and G of triangle EFG approach B, and p approaches the altitude BE of triangle ABC which can be shown to be $2\cos\alpha(\cos\alpha-\sin\alpha)$ in the limiting case. Therefore $p=2\cos\alpha(\cos\alpha-\sin\alpha)$ at $\phi=\pi/2+\alpha$ and, by symmetry, it will have the same value at $\phi=\pi/2-\alpha$. But it can be shown that

$$2\cos\alpha(\cos\alpha-\sin\alpha)\leqq\frac{4\cos^3\alpha}{1+\sin\alpha},$$

where equality holds only for $\alpha=\pm\pi/2$. Therefore $p(\pi/2)=4\cos^3\alpha/(1+\sin\alpha)$ must be the maximum value of the function $p(\phi)$. Thus we have proved that p attains its maximum when $AB=AC$.

Finally, it must be proved that among all the isosceles triangles ABC of a given perimeter l, the ratio p/l attains its maximum when ABC is equilateral. The analytical proof of this property is straightforward and will be omitted here.

This completes the proof of the main result of this paper.

I wish to acknowledge my indebtedness to Professor N. A. Court of the University of Oklahoma and to Professor James D. Monk of the University of Colorado for their valuable suggestions.

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THE STEREOGRAPHIC PROJECTION IN VECTOR NOTATION

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The stereographic projection is usually presented by either trigonometry or complex variables. In this paper vectors are used to derive a few of the well-known properties of the stereographic projection of the sphere.

In Figure 1 the point P on the sphere has the stereographic image Q in the tangent plane at N. We will determine the transformation equations relating points of the sphere $P(x, y, z)$ and points of the plane $Q(u, v)$ from the fact that this is equivalent to determining the relation between the vectors $\mathbf{R}$ and $\mathbf{E}$. For simplicity it is assumed that the u-v axes are set parallel to the respective x-y axes and that the sphere is of unit radius.

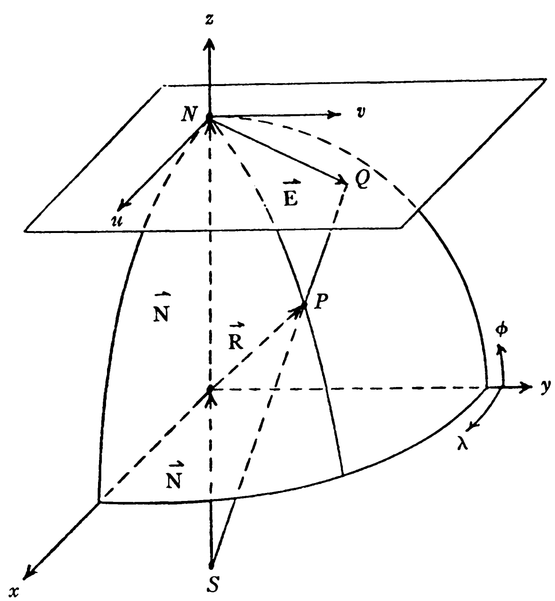

FIG. 1. Stereographic Projection of P onto a plane tangent at the north pole N.

With the aid of Fig. 1, it follows that $\mathbf{E}=t_1(\mathbf{N}+\mathbf{R})-2\mathbf{N}$, where the scalar t_1 must satisfy $\mathbf{E}\cdot\mathbf{N}=[t_1(\mathbf{N}+\mathbf{R})-2\mathbf{N}]\cdot\mathbf{N}=0$. Since $\mathbf{R}\cdot\mathbf{R}=\mathbf{N}\cdot\mathbf{N}=1$,

$$t_1 = \frac{2}{1+\mathbf{R}\cdot\mathbf{N}}$$

and

$$\mathbf{E} = \frac{2}{1+\mathbf{R}\cdot\mathbf{N}}[\mathbf{R}-(\mathbf{R}\cdot\mathbf{N})\mathbf{N}]. \tag{1}$$

In a similar manner the inverse of (1); i.e., an explicit equation for $\mathbf{R}$ in terms of $\mathbf{E}$ and $\mathbf{N}$, is obtained from $\mathbf{R}\cdot\mathbf{R}=[(2\mathbf{N}+\mathbf{E})t_2-\mathbf{N}]\cdot[(2\mathbf{N}+\mathbf{E})t_2-\mathbf{N}]=1$, where $t_2>0$ is a scalar. The result is

$$\text{(2)} \qquad \mathbf{R} = \frac{1}{4 + \mathbf{E}\cdot\mathbf{E}} [4\mathbf{E} + (4 - \mathbf{E}\cdot\mathbf{E})\mathbf{N}].$$

From equation (1) it can be shown that a straight line or a circle in the image plane given by $a\mathbf{E}\cdot\mathbf{E}+2\mathbf{B}\cdot\mathbf{E}+c=0$, where a and c are arbitrary scalars and $\mathbf{B}$ is an arbitrary vector in the image plane, corresponds to a circle on the sphere. Thus, the equation of the straight line or circle (depending upon whether or not $a=0$) in the image plane maps into the plane curve on the sphere $\mathbf{R}\cdot[4\mathbf{B}+(c-4a)\mathbf{N}]+4a+c=0$ which, consequently, must be a circle.

To prove that the stereographic projection is conformal we will show that the differentials $d\mathbf{E}\cdot d\mathbf{E}$ and $d\mathbf{R}\cdot d\mathbf{R}$ are proportional. Directly from (1) we obtain

$$d\mathbf{E} = \frac{2}{(1 + \mathbf{R}\cdot\mathbf{N})^2} [(1 + \mathbf{R}\cdot\mathbf{N})d\mathbf{R} - d\mathbf{R}\cdot\mathbf{N}(\mathbf{R} + \mathbf{N})],$$

and, since $\mathbf{R}\cdot\mathbf{R}=1$ implies $\mathbf{R}\cdot d\mathbf{R}=0$, it follows that

$$d\mathbf{E}\cdot d\mathbf{E} = \frac{4}{(1 + \mathbf{R}\cdot\mathbf{N})^2} d\mathbf{R}\cdot d\mathbf{R}.$$

By expressing $\mathbf{E}$ and $\mathbf{R}$ in terms of components; viz, $\mathbf{E}=(u, v, o)$, $\mathbf{R}=(x, y, z)$ and $\mathbf{N}=(0, 0, 1)$, we obtain the following standard transformation equations for the polar stereographic projection:

$$\text{(3)} \qquad \mathbf{E}(u, v) = \frac{2}{1 + z} (x, y)$$

$$\text{(4)} \qquad \mathbf{R}(x, y, z) = \frac{1}{4 + u^2 + v^2} (4u, 4v, 4 - u^2 - v^2).$$

Since $\mathbf{R}(x, y, z)=(\cos\phi \sin\lambda, \cos\phi \cos\lambda, \sin\phi)$, where $-\pi/2\leqq\phi\leqq\pi/2$ and $0\leqq\lambda\leqq 2\pi$, equation (3) can also be written as

$$\text{(5)} \qquad \mathbf{E}(u, v) = \frac{2\cos\phi}{1 + \sin\phi} (\sin\lambda, \cos\lambda).$$

Although the transformation equations of the stereographic projection were derived for the specific case where the plane is tangent to the sphere at the north pole, these equations will also be valid for any other tangent plane if the components of the vectors $\mathbf{R}$, $\mathbf{E}$, and $\mathbf{N}$ are referred to appropriate coordinate systems. To this end, in each tangent plane we attach a right-hand rectangular coordinate system $u'v'$ that (1) originates at the point of tangency N' and (2) is oriented so that, except for the poles, the u'-axis points east and the v'-axis points north (Fig. 2). At the poles we may choose any orientation for the u'- and v'-axes as, indeed, we have already done for the north pole. Also, we set up a right-hand rectangular coordinate system $x'y'z'$ at the center of the sphere oriented so that the x'- and y'-axes are parallel, respectively, to the u'- and v'-axes. Equations (3) and (4) must be valid in these coordinate systems, since the x'-y'-z' axes are related to the u'-v' axes (Fig. 2) in the same manner as the

x-y-z axes are related to the u-v axes (Fig. 1). Problems involving various image planes can, therefore, be reduced to a rotation of the x-y-z axes into x'-y'-z' axes.

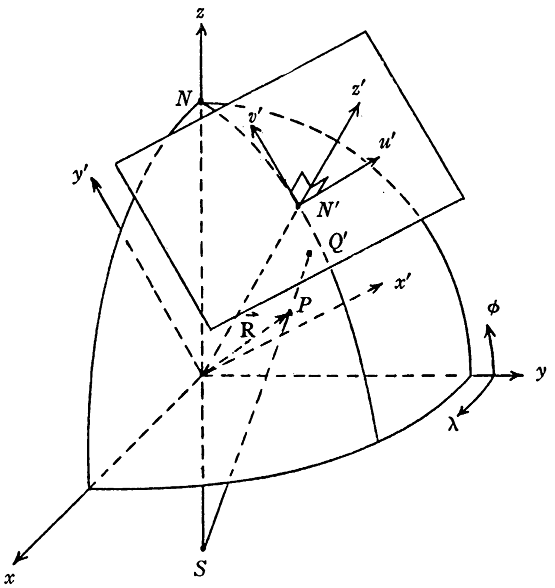

FIG. 2. Stereographic Projection of P onto a plane tangent at N'.

As an example (Fig. 2), we now transform the point $P(\phi, \lambda)$ on the sphere into its stereographic image $Q'(u', v')$ in the plane tangent to the sphere at $N'(\phi_o, \lambda_o)$. First, compute the $x'y'z'$ components of **R** by the orthogonal transformation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -\cos\lambda_o & \sin\lambda_o & 0 \\ -\sin\phi_o \sin\lambda_o & -\sin\phi_o \cos\lambda_0 & \cos\phi_o \\ \cos\phi_o \sin\lambda_o & \cos\phi_o \cos\lambda_o & \sin\phi_o \end{bmatrix} \begin{bmatrix} \cos\phi \sin\lambda \\ \cos\phi\cos\lambda \\ \sin\phi \end{bmatrix}.$$

Then, by substituting the components $x'y'z'$ into equation (3), we obtain the following well-known transformation equations of the stereographic horizon projection of the sphere:

$$u' = \frac{2\cos\phi\sin(\lambda_o - \lambda)}{1 + \sin\phi_o \sin\phi + \cos\phi_o\cos\phi\cos(\lambda_o - \lambda)},$$

$$v' = \frac{2[\cos\phi_o \sin\phi - \sin\phi_o\cos\phi\cos(\lambda_o - \lambda)]}{1 + \sin\phi_o \sin\phi + \cos\phi_o\cos\phi\cos(\lambda_o - \lambda)}.$$

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A MAXIMAL GENERALIZATION OF FERMAT'S THEOREM

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1. Introduction. It is well known that there are two equivalent forms of Fermat's Theorem: for any prime p,

I. $(a, p) = 1$ implies $a^{p-1} \equiv 1 \pmod{p}$.

II. for all a, $a^p \equiv a \pmod{p}$.

Classically, the generalizations of Fermat's Theorem have been generalizations of form I. These are:

EULER'S THEOREM. *For any integer m,*

III. $(a, m) = 1$ *implies* $a^{\phi(m)} \equiv 1 \pmod{m}$.

CARMICHAEL'S THEOREM [1, 2, 3, 4]. *For any integer m,*

IV. $(a, m) = 1$ *implies* $a^{\lambda(m)} \equiv 1 \pmod{m}$.

($\lambda(m)$ will be defined shortly). Moreover, Carmichael has shown that IV is a best possible theorem, i.e., a maximal generalization of I, by showing that there always exists a "primitive λ-root," that is, an element whose order (mod m) is actually $\lambda(m)$.

Some attention has been given to the problem of generalizing II. The following is attributed to Rédei in [5] and [6].

V. for all a, $a^m \equiv a^{m-\phi(m)} \pmod{m}$.

A different kind of generalization is that which was first stated by Gauss [7, p. 84, 6, 12].

VI. for all a, $\sum_{d|n} \mu(d)a^{n/d} \equiv 0 \pmod{n}$.

Here, μ is the Mobius function.

I have discovered the following maximal generalization of II, corresponding to IV.

THEOREM 1. *Let* $m = p_1^{e_1}p_2^{e_2} \cdots p_n^{e_n}$, *and define* $N(m) = \max(e_1, e_2, \cdots, e_n)$. *Then, for any integer m,*

VII. for all a, $a^{\lambda(m)+N(m)} \equiv a^{N(m)} \pmod{m}$.

I shall show that VII gives the congruence of lowest degree of the form: for all a, $a^r = a^s \pmod{m}$, $r \neq s$. Some information on the size of $\lambda(m)+N(m)$ will be obtained and V and VI will be shown to follow easily from these results. Some applications to polynomial functions (mod m) will be given.

Proof of the theorem. Let $m = p_1^{e_1}p_2^{e_2} \cdots p_n^{e_n}$, and let $m_i = p_i^{e_i}$. Then $\lambda(m)$ is defined for all integers by the following: $\lambda(p^e) = \phi(p^e)$, except that $\lambda(2^e) = \frac{1}{2}\phi(2^e)$, whenever $e \geqq 3$; $\lambda(m) = [\lambda(m_1), \lambda(m_2), \cdots, \lambda(m_n)]$ where the brackets indicate the *LCM*.

In order to show that VII holds for a, it suffices to show that, for all i,

$$a^{\lambda(m)+N(m)} \equiv a^{N(m)} \pmod{m_i}. \tag{1}$$

To do this, we rewrite (1) in the following form:

$$a^{N(m)}(a^{\lambda(m)} - 1) \equiv 0 \pmod{m_i}. \tag{2}$$

If $p_i \nmid a$, then $(a, p_i) = (a, m_i) = 1$ and we have $a^{\lambda(m_i)} - 1 \equiv 0 \pmod{m_i}$ by IV. Since $\lambda(m_i) | \lambda(m)$, we have $m_i | (a^{\lambda(m_i)} - 1) | (a^{\lambda(m)} - 1) | a^{N(m)}(a^{\lambda(m)} - 1)$, so (2) holds.

If $p_i | a$, then since $N(m) \geqq e_i$, we have

$$m_i = p_i^{e_i} | a^{e_i} | a^{N(m)} | a^{N(m)}(a^{\lambda(m)} - 1),$$

and so (2) holds and the theorem holds.

2. Further results.

THEOREM 2. *Let $r > s$; then $a^r \equiv a^s \pmod{m}$ for all a, if and only if $\lambda(m) | r - s$ and $s \geqq N(m)$.*

Proof. If $\lambda(m) | r - s$ and $s \geqq N(m)$, then a slight change in the proof of Theorem 1 shows that $a^r \equiv a^s \pmod{m}$ for all a.

For the converse, we use Carmichael's result, mentioned above, that there is an element, say b, such that the order of $b \pmod{m}$ is precisely $\lambda(m)$. Since $b^r \equiv b^s \pmod{m}$, we must have that $\lambda(m) | r - s$. Recalling the expression for the prime factorization of m, let $m_0 = p_1 p_2 \cdots p_n$. Then the quantities $m_0, m_0^2, m_0^3, \cdots, m_0^{N(m)} (\equiv 0 \pmod{m})$ are all distinct $\pmod{m}$ and all powers m_0^k, $k \geqq N(m)$ are $\equiv 0 \pmod{m}$. Hence two different powers of m_0 are congruent $\pmod{m}$ if and only if the smaller power is $\equiv 0 \pmod{m}$, i.e., iff the smaller exponent s, satisfies $s \geqq N(m)$.

Theorem 2 states precisely the sense in which Theorem 1 is a maximal generalization of II. From Theorem 2, the minimal possible values of r and s are $s = N(m)$ and $r = \lambda(m) + N(m)$. Theorem 1 shows that these minimal values work.

THEOREM 3. *$\lambda(m) + N(m) \leqq \phi(m) + N(m) \leqq m$, with simultaneous equality iff m is prime or $m = 4$.*

Proof. The left hand inequality is trivial since $\lambda(m) | \phi(m)$, and equality is known to hold iff m has a primitive root, i.e., for q^e, $2q^e$, 2, and 4, where q is any odd prime.

We designate the residue class ring $\pmod{m}$ as J_m. Then $\phi(m)$ is the number of elements of J_m which are prime to m, so the right hand inequality will be established if we demonstrate that there are at least $N(m)$ elements of J_m which are not prime to m. The above mentioned elements $m_0, m_0^2, \cdots, m_0^{N(m)}$ are distinct $\pmod{m}$ and are certainly not prime to m. Equality can hold on the right iff these numbers are all the elements of J_m which are not prime to m. But p_i is not prime to m and the only way p_i can be congruent to a power of m_0 is if

$p_i = m_0$ and this must hold for all i. That is, m must be a prime power, say $m = p^e$.

Thus $p, p^2, \cdots, p^e$ are $e(=N(m))$ distinct elements of J_m which are not prime to m. If $e=1$, then $p=p^e\equiv 0$ is the only such element and equality holds. Now suppose $e\geqq 2$. Then the number $2p$ will be such an element which is not included among $p, p^2, \cdots, p^e$ unless $2p=p^2$, i.e., $p=2$. But if $p=2$ and $e\geqq 3$, then $3p=6$ will be an element of J_m which is not prime to m and which is not one of $2, 2^2, 2^3, \cdots, 2^e$. If $p=2$ and $e=2$, then equality holds, so we have equality on the right iff m is prime or $m=4$. Since equality on the right implies equality on the left, we are done.

Alternatively, once one has seen that $m=p^e$, the theorem may be completed by noting that $m-\phi(m)=p^{e-1}\geqq e=N(m)$ with equality iff m is prime or $m=4$.

COROLLARY 1. *For all m and all a, we have the following*:

V. $$a^m \equiv a^{m-\phi(m)} \equiv a^{m-\lambda(m)} \pmod m; \qquad a^{N(m)+\phi(m)} \equiv a^{N(m)} \pmod m.$$

(This last congruence is mentioned by Nielsen [11].) These follow directly from Theorems 1, 2, and 3.

COROLLARY 2. *For all m*,

VI. $$\text{for all } a, \qquad \sum_{d|m} \mu(d)a^{m/d} \equiv 0 \pmod m.$$

Proof. We have

$$\sum_{d|m} \mu(d)a^{m/d} = \prod_{p^e||m} (a^{p^e} - a^{p^{e-1}}),$$

as is easily seen by expanding the product. By V, for the case $m=p^e$, we have that

$$a^{p^e} \equiv a^{p^{e-1}} \pmod{p^e}.$$

Hence p^e divides the product for every $p^e||m$; thus m divides the product and we are done.

Gegenbauer [12] has used this fact to show that the same conclusion holds if the function μ is replaced by any other function F such that $\sum_{d|m} F(d) \equiv 0 \pmod m$ for all m. The case $F=\phi$ gives an interesting result of MacMahon [13].

We note the following properties of $N(m)$.

$$a\,|\,b \quad \text{implies} \quad N(a) \leqq N(b)$$
$$\max(N(a), N(b)) \leqq N(ab) \leqq N(a) + N(b)$$
$$(a, b) = 1 \quad \text{implies} \quad N(ab) = \max(N(a), N(b)).$$

3. Applications to polynomial functions (mod M).

LEMMA. *There are at most $m^{\lambda(m)+N(m)}$ distinct polynomial functions* (mod m).

Proof. Let f be any polynomial function (mod m) and let $F(x)$ be any polynomial in $J_m[x]$ which represents f, that is, such that $f(a)\equiv F(a)$ (mod m) for all a. Let $B(x)=x^{\lambda(m)+N(m)}-x^{N(m)}$. By a modified form of the division algorithm [8], we can write $F(x)\equiv B(x)Q(x)+R(x)$, where $\deg(R)<\deg(B)=\lambda(m)+N(m)$, or $R\equiv 0$. By Theorem 1, $f(a)\equiv F(a)\equiv R(a)$ (mod m) for all a, so that every polynomial function (mod m) is represented by a polynomial in $J_m[x]$, which has degree $<\lambda(m)+N(m)$ or is 0. There are $m^{\lambda(m)+N(m)}$ such polynomials so we are done.

COROLLARY. *Every function from J_m to J_m is a polynomial function if and only if m is prime.*

Proof. By Lagrange's Interpolation Formula, every function over a finite field, such as J_p, is a polynomial function. Conversely, since there are m^m functions: $J_m\rightarrow J_m$, the Lemma and Theorem 3 show that there are fewer polynomial functions than functions for all composite m other than 4. But one can verify that $2x^2-2x\equiv 2x^3-2x^2\equiv 2x^3-2x\equiv 0$ (mod 4) and that these are all the representations of the zero function as a polynomial of degree less than 4. From this it follows that there are exactly $(1/4)4^4=4^3(<4^4)$ polynomial functions (mod 4).

This corollary is a special case of the following more general Theorem 4. This theorem is known in a much wider context [9], but it does not seem to be generally known. I include it here, partly in the hope of making it more widely known. The proof is essentially that of [9].

THEOREM 4. *Let R be a commutative ring with identity. Then every function from R to R is a polynomial function iff R is a finite field.*

Proof. If R is a finite field, then every function is a polynomial function by Lagrange's Interpolation Formula. Conversely, first suppose R is infinite, of cardinality r. The cardinality of $R[x]$ is also r, but that of R^R (the set of all functions: $R\rightarrow R$) is $r^r\geqq 2^r>r$, so R cannot have the desired property. Now suppose that R is not a field. Then R must have a proper ideal A since a commutative ring with identity is a field iff it has no proper ideals. Since A is proper, we can pick $a\neq 0$, $a\in A$, and $b\notin A$. Consider the function f such that $f(a)=b$ and $f(x)=0$ for $x\neq a$. No polynomial can represent this function since $a\equiv 0$ (mod A) implies $F(a)\equiv F(0)$ (mod A) for any polynomial $F(x)$.

Historical notes. Since I wrote the first draft of this paper, I have become aware that this work has been anticipated and even stated by others. Lucas [1] stated, without proofs, IV, Theorem 1 and that $\lambda(m)+N(m)$ "est toujours plus petit que" m. There is an apparent typographic error in that he gives $N(m)$ as "le plus petit exposant."

Dickson [7, p. 78, item 110] corrects this last error but interprets Lucas as asserting $\lambda(m)+N(m)<m$. He then [7, p. 81, item 129; p. 82, item 143] states that Bachmann [10] proved IV, Theorem 1 and cases of Theorem 3. However, Bachmann's concept of $\lambda(m)$ does not account for the factor $\frac{1}{2}$ when 8 divides m. Hence Dickson is incorrect. Nonetheless, Bachmann's work is sufficiently valid that I have patterned the present proof of Theorem 1 after Bachmann, thereby greatly simplifying the proof from what I had originally.

Carmichael [2] was the first to prove IV. He introduced the symbol $\lambda(m)$ and noted the difference between $\lambda(m)$ and the version used by Bachmann. He must have been unaware of Lucas' statements, otherwise he would have proven them.

Remarks. One might ask if Theorem 1 is a best possible theorem in the sense that $x^{\lambda(m)+N(m)}-x^{N(m)}$ gives the monic polynomial of lowest degree which represents 0 (mod m). The answer to this is "no".

Rédei and Szele [14] have shown that Theorem 4 holds without the assumption that R has an identity. It appears to be an open question whether commutativity can be eliminated even if the identity is retained.

Purely algebraic analogs of Theorems 1, 2, and 3 exist in any finite commutative ring with identity. Schwarz [5] has already found an analog of V for finite semigroups and I feel that the actual content of Theorems 1, 2, and 3 is an algebraic rather than a number-theoretic content.

I would like to thank Dr. Steven J. Bryant for posing the problem which led me to these results and to thank the referee for the references to [5] and [6].

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THE REAL SOLUTIONS OF $x^y=y^x$

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Let S be the set of points (x, y) in the euclidean plane where x and y satisfy

$$x^y = y^x \tag{1}$$

(in the sense that there exists one value α of x^y and one value β of y^x with $\alpha=\beta$). In [1] it is proved that S is contained in the set

$$\{(x, y): x \log |y| = y \log |x|\},$$

this latter set consisting of the union of five curves, one of which is the line $y=x$. The four remaining curves, say C_1, C_2, C_3, and C_4 are shown to lie in the first, second, third and fourth quadrants respectively.

THEOREM 1 ([1]). *In the above situation,*

(a) *any point of C_1 is a point of S,*

(b) *the points of S are everywhere dense on C_2, C_3, and C_4 and*

(c) $(-1, 1)\in C_2$, $(1, -1)\in C_4$ *and these two points are not elements of S.*

The following questions are then raised.

(a) Are there any points of C_3 which are not points of S?

(b) Are there any points of C_2 or C_4, apart from those given in Theorem 1 (c), which are not points of S?

In this paper we shall prove the following result and thus answer both questions.

THEOREM 2. *The set S is countably dense on C_2, C_3, and C_4. Further, if $(x, y)\in S-C_1$ then x and y are both rational or are both irrational.*

The following question is left unanswered: Does there exist a rational number x and an irrational number y such that $(x, y)\in S$? If so then clearly $(x, y)\in C_1$.

Proof of Theorem 2. This depends on the following two lemmas the first of which is well known (e.g. [2] p. 47).

LEMMA 1. *Every uncountable set contains a perfect subset.*

LEMMA 2. *Suppose that $f(x)$ and $g(x)$ are differentiable on some interval I and that $f(x)=g(x)$ on some perfect subset Q of I. Then $f'(x)=g'(x)$ on Q.*

Proof. Let $h(x)=f(x)-g(x)$. Then if $x\in Q$

$$h'(x) = \lim_{x'\to x} \frac{h(x') - h(x)}{x' - x}$$

exists and this limit may be evaluated by restricting x' to Q. Since $x, x'\in Q$ we have

$$h(x') = h(x) = 0$$

and so $h'(x)=0$.

By the definition of exponents (1) is equivalent to

$$\exp [x \log y] = \exp [y \log x]. \tag{2}$$

Writing $x = |x| \exp [i\theta]$, $y = |y| \exp [i\phi]$ we see that x and y satisfy (2) if and only if there exist integers p, q, and n such that

$$x(\log |y| + i\theta + 2\pi p i) = y(\log |x| + i\phi + 2\pi q i) + 2\pi n i. \tag{3}$$

Since x and y are real (and nonzero), θ and ϕ can only assume the values 0 or π and these values are taken according to the quadrant in which (x, y) is situated. Equation (3) thus becomes

$$x \log |y| = y \log |x| \tag{4}$$

together with

$$px = qy + n \tag{4.1}$$

$$(2p + 1)x = 2qy + 2n \tag{4.2}$$

$$(2p + 1)x = (2q + 1)y + 2n \tag{4.3}$$

$$px = (2q + 1)y + 2n \tag{4.4}$$

in the sense that the points (x, y) of S which lie in the kth quadrant must satisfy (4) and (4.k) ($k = 1, \cdots, 4$). It is to be emphasized that x and y satisfy (4.k) if and only if there exist integers p, q and n such that x, y, p, q and n together satisfy (4.k).

In the case of (4.1) we may take $p = q = n = 0$ and hence every pair x, y is a solution of this equation. This implies that the points of S in the first quadrant are determined solely by the solutions of (4) and this is Theorem 1(a).

The line $y = x$ is obtained by taking $p = q = 1$ and $n = 0$ in equations (4.1) and (4.3).

The solutions of equations (4.2), (4.3) and (4.4) yield a countable family L of straight lines none of which are degenerate since at least one coefficient is nonzero. The first part of Theorem 2 will thus be proved if we can show that any straight line (and hence any element of L) intersects the curves C_2, C_3 and C_4 in a countable set. Since C_j is geometrically similar to $C_{j+2}(j = 1, 2)$ (see [**1**]) it suffices to prove this result for C_1 and C_2.

The proof for C_1. Since C_1 lies in the first quadrant, equation (4) becomes

$$x \log y = y \log x, \qquad x, y > 0. \tag{5}$$

It is shown in [**1**] that the only point of C_1 which lies on the line $y = x$ is the point (e, e), $e = \exp [1]$. By the Implicit Function Theorem (e.g. [**3**] p. 326) we see that $y'(x)$ exists except possibly at $x = e$. Differentiation of (5) yields

$$y'(x) = \frac{s^2(t - 1)}{t^2(s - 1)}, \qquad (s \neq 1) \tag{6}$$

where $t = \log x$ and $s = \log y$. The point (e, e) (equivalently the point $s = t = 1$) is not a singular point of C_1, the expression given in (6) merely has a removable singularity at this point. Differentiating (6) and rewriting (5) in the form

$$s \exp [t] = t \exp [s] \tag{7}$$

we see that

$$y''(x) = \frac{s^3[(2-t)(s-1)^2 + (s-2)(t-1)^2]}{t^4(s-1)^3 \exp [s]}$$

and thus $y''(x) = 0$ if and only if

$$(2-t)(s-1)^2 = (2-s)(t-1)^2,$$

that is, if and only if

$$3 + st = 2(s+t). \tag{8}$$

Since y is a strictly decreasing function of x for $(x, y) \in C_1$ (see [**1**]) any line of the form $x =$ constant has at most one intersection with C_1. Suppose now that some line, say

$$y = mx + c$$

intersects C_1 in an uncountable set Q. Then $Q - \{(e, e)\}$ is uncountable and, by Lemma 1, contains a perfect subset P. Thus if $(x, y) \in P$ we have

$$y(x) = mx + c.$$

Applying Lemma 2 twice we see that for $(x, y) \in P$,

$$y''(x) = 0,$$

and hence (8) is valid for all $(x, y) \in P$. Using (8) we can express s as a function of t and then substitution in (7) gives

$$(2t-3) \exp [t] = t(t-2) \exp [(2t-3)/(t-2)]. \tag{9}$$

Writing

$$F(t) = (2t-3) \exp [t]$$

and

$$G(t) = t(t-2) \exp [(2t-3)/(t-2)]$$

we see from (9) that

$$F(t) = G(t) \tag{10}$$

on some uncountable set T of values of t. Applying Lemma 2 we deduce that

$$F'(t) = G'(t) \tag{11}$$

for $t \in T$. Eliminating the exponential terms from (10) and (11) we conclude that t must be the solution of a quartic equation and this contradicts the cardinality of T. The result is thus proved for C_1.

The proof for C_2. It is sufficient to prove the result for the reflection of C_2 in the y-axis, that is the curve C_2^* given by

$$x \log y = - y \log x, \qquad x, y > 0.$$

An analogous argument to that used above will establish the result for C_2^*, the only difference being as follows. The Implicit Function Theorem is used to deduce that the intersections of a line $x = \text{constant}$ with the curve C_2^* are isolated and hence countable.

Finally if $(x, y) \in S - C_1$ then either $x = y$ or (x, y) is on one of the curves C_2, C_3 or C_4. Thus $x = y$ or x and y must satisfy one of the equations (4.2), (4.3) and (4.4). Thus x is rational if and only if y is rational.

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ANSWERS

A 378. (1) Analytic solution.

$$\overline{A} = \frac{1}{abc} \int_0^a \int_0^b \int_0^c \{A - \tfrac{1}{2}[z(b - y) \sin A + x(c - z) \sin B + y(a - x) \sin C]\} dxdydz,$$

$\overline{A} = A/4$ where A is the area of the given triangle.

(2) Geometric solution. If a series of triangles have a common base and their vertices be in a given finite straight line which is wholly on the same side of the base, the average of all triangles thus formed is that whose vertex is at the middle of the line segment; since for every triangle which exceeds this, there is obviously another just as much less than it. Consequently the mean-inscribed triangle is the one joining the midpoints of the sides, and $\overline{A} = A/4$.

A 379. (a) Let the circumcircle of BCG cut FJ at O. Then since angle $OCG = 120°$ we have $OC = BC - CG$. So

$$OJ = BC - CG + 2CG = BC + CG = FJ/2.$$

(b) The chords BO, OG and GB are equal because the angles OCB, GBO and BOG are equal.

A 380. Consider $(\sqrt{3})^{\sqrt{2}}$. If this representation is rational, we are finished. If not, then the above representation is irrational. Then consider

$$[(\sqrt{3})^{\sqrt{2}}]^{\sqrt{2}} = 3.$$

A 381. Let d be the unknown side. In a circumscribed quadrilateral for any two opposite sides (e.g., for a and c) we have

$$a = s - c$$

where

$$s = \tfrac{1}{2}(a + b + c + d).$$

Thus

$$\tfrac{1}{2}(a + b + c + d) - d = b$$

so

$$d = a + c - b.$$

The area of an inscribed quadrilateral is known to be

$$T = \sqrt{\{(s - a)(s - b)(s - c)(s - d)\}}.$$

Substituting, we have

$$T = \sqrt{abcd}$$

where $d = a + c - b$.

A 382. As viewed in the plane, the object will be accelerated toward the center (i.e., the z-axis) with acceleration

$$g \frac{dz}{dr}$$

which is

$$ag/r^2.$$

Hence it will exhibit an inverse-square planetary orbit: ellipse, hyperbola or parabola.

(Quickies on page 134)

THE RELATION OF $f_+'(a)$ TO $f'(a+)$

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1. Introduction. Advanced Calculus textbooks are usually careful to point out the distinction between the *derivative on the right*, defined as

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}, \tag{1}$$

and the *right-hand limit of the derivative*, denoted by $f'(a+)$ and defined by

$$f'(a+) = \lim_{x \to a^+} f'(x).$$

A slight generalization of (1) is the *right-hand derivative* denoted by $f'_+(a)$ and defined by

$$f_+'(a) = \lim_{x \to a^+} \frac{f(x) - f(a+)}{x - a}.$$

To emphasize the distinction, it is customary to cite an example such as $f(x) = x^2 \sin x^{-1} (x \neq 0)$, for which $f'_+(0) = 0$ whereas $f'(x)$ oscillates with amplitude 1 in every neighborhood of $x = 0$, so that $f'(0+)$ does not exist. Other examples are set out in the accompanying table.

From a pedagogical viewpoint, it might be preferable to exhibit a function for which $f'_+(a)$ and $f'(a+)$ both exist but are different finite quantities. An elementary example which exhibits this behavior is given in section 2. There is, however, a somewhat pathological character about the function: in every open interval with a as left end point, there are infinitely many points at which $f'(x)$ does not exist. From a strictly classical standpoint it would appear that $f'(a+)$ does not exist in such a case, for the traditional epsilon-delta definition of right-hand limit ($\lim_{x \to a^+} F(x) = A$ if for every positive number δ there exists a positive number ϵ such that $|F(x) - A| < \delta$ whenever $a < x < a + \epsilon$) involves the tacit assumption that the function under consideration is defined throughout some open interval to the right of a. Taking a more modern view, one would relate the limit concept to the subspace topology and define:

If F is defined on a set S of real numbers and a is a limit point of $S \cap \{x \mid x > a\}$, then $\lim_{x \to a^+} F(x) = A$ if for every positive number δ there exists a positive number ϵ such that $|F(x) - A| < \delta$ whenever $x \in S \cap \{x \mid a < x < a + \epsilon\}$.

The example in section 2 is based upon this definition of limit. Indeed, the construction of such an example depends critically upon a being a limit point of "holes" in the domain of definition of $f'(x)$. In section 3 we show that if this is not the case, then the simultaneous existence of $f'_+(a)$ and $f'(a+)$ implies their equality. This conclusion is reached through a series of corollaries to a more general result: if $f(a+)$ and $f'_+(a)$ both exist—the latter quantity not necessarily finite—and $f'(x)$ exists on some interval (a, b), then $f'(x)$ cannot be bounded away from $f'_+(a)$ on any interval having a as its left end point. Indeed, it is an immediate consequence of the Fundamental Theorem of Calculus that, as $x \to a^+$, $f'(x)$ converges to $f'_+(a)$—at least in mean value, i.e.,

$$\lim_{x \to a^+} \frac{1}{x - a} \int_a^x f'(x) dx = f_+'(a).$$

In section 4 we relax the assumption that $f'_+(a)$ exists. If f is differentiable on (a, b) and $f'(a+)$ is finite, then the existence of both $f(a+)$ and $f'_+(a)$ is assured; moreover, $f'_+(a) = f'(a+)$. If $f'(x)$ oscillates with finite amplitude as $x \to a^+$, $f(a+)$ exists, and $(x - a)^{-1}[f(x) - f(a+)]$ does not become unbounded as

$x \to a^+$. Finally, the convergence of $f'(x)$ in the mean-value sense as $x \to a^+$ is shown to be sufficient as well as necessary for the existence of $f'_+(a)$.

The results of sections 3 and 4 explain the missing entries in the table: they describe circumstances which don't happen.

$f'(0+)$ / $f'_+(0)$	finite	∞	$-\infty$	bounded oscillation	unbounded oscillation
finite	x			$x^2 \sin \frac{1}{x}$	$x^2 \sin \frac{1}{x^2}$
∞		$\sqrt{x}$			$x^2 \sin \frac{1}{x^2} + \sqrt{x}$
$-\infty$			$-\sqrt{x}$		$x^2 \sin \frac{1}{x^2} - \sqrt{x}$
bounded oscillation				$x \sin \ln x$	$x \sin \frac{1}{x}$
unbounded oscillation					$\sqrt{x} \sin \frac{1}{x}$
$f(0+)$ does not exist		$\ln x$	$\frac{1}{x}$		$\sin \frac{1}{x}$

Table of functions differentiable on (0, 1) with specified behavior of $f'_+(0)$ and $f'(0+)$.

By modifying the theorems in an obvious manner, one could derive a series of results relating the left-hand derivative to the left-hand limit of the derivative.

Certain of the results given here are well known. See, for example, [1]. For continuity of presentation, they are absorbed into the general treatment.

2. A function for which $f'_+(0)$ and $f'(0+)$ are different finite quantities. For all real x in the interval $(0, 1/2]$, we define

$$f(x) = [x^{-1}]^{-1}; \tag{2}$$

where $[X]$ denotes the greatest integer not greater than X. Clearly, $f(0+)=0$ and $f'(x)$ vanishes throughout its domain of definition, viz.,

$$D \overset{\text{def.}}{=} \{x \mid 0 < x < \tfrac{1}{2};\ x \neq \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \cdots\}.$$

Since $x=0$ is a limit point of D, $f'(0+)=0$.

We determine $f'_+(0)$ by bounding $x^{-1}f(x)$ between two functions which approach the same limit as $x \to 0^+$. The definition (2) is equivalent to the statement:

$$\forall x \in \left(\frac{1}{n+1}, \frac{1}{n}\right], f(x) = \frac{1}{n} \quad (n = 2, 3, 4, \cdots).$$

Now, it can be verified by elementary methods that

$$\forall x \in \left(\frac{1}{n+1}, \frac{1}{n}\right], \frac{x}{1-x} > \frac{1}{n} \quad (n = 2, 3, 4, \cdots)$$

and by tautology that

$$\forall x \in \left(\frac{1}{n+1}, \frac{1}{n}\right], x \leqq \frac{1}{n} \quad (n = 2, 3, 4, \cdots).$$

These three statements taken together imply that

$$\forall x \in \left(\frac{1}{n+1}, \frac{1}{n}\right], x \leqq f(x) < \frac{x}{1-x} \quad (n = 2, 3, 4, \cdots). \tag{3}$$

However,

$$\left(0, \frac{1}{2}\right] = \bigcup_{n=2}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right].$$

Thus, since x is positive on $(0, 1/2)$, the statement (3) implies that

$$\forall x \in \left(0, \frac{1}{2}\right], 1 \leqq \frac{f(x)}{x} \equiv \frac{f(x) - f(0+)}{x - 0} < \frac{1}{1-x},$$

and it follows immediately that $f'_+(0) = 1$.

3. Results which follow from the existence of $f'_+(a)$. If f is differentiable on an open interval having a as left end point, the existence of $f'_+(a)$ imposes a strong constraint on the behavior of $f'(x)$ as x approaches a from the right. The main result can be expressed as a theorem:

If

(i) *$f(a+)$ exists and is finite;*

(ii) *$f'(x)$ exists on some open interval (a, b);*

(iii) *there exist quantities* M *and* N (*the possibilities* $M=-\infty$ *and* $N=+\infty$ *are not excluded, nor is their simultaneous occurrence*) *such that* $M<f'(x)<N$ *for every* x *in* (a, b); *then either* $M\leqq f'_+(a)\leqq N$ *or* $f'_+(a)$ *does not exist.*

Proof. There is no loss of generality in assuming that f is continuous on the right at $x=a$: the existence of $f(a+)$ implies that any right-hand discontinuity of f at a is removable.

By hypothesis (ii), f is differentiable, hence continuous, on (a, b). Consider any x in (a, b). Since f is continuous on $[a,x]$ and differentiable on (a, x), the Law of the Mean applies: there exists some point X in (a, x) such that

$$\frac{f(x)-f(a+)}{x-a}=f'(X).$$

By hypothesis (iii), therefore,

$$M<\frac{f(x)-f(a+)}{x-a}<N$$

for every x in (a, b). Hence

$$M\leqq \lim_{x\to a^+}\frac{f(x)-f(a+)}{x-a}\leqq N$$

if the indicated limit exists. If it does exist, it is $f'_+(a)$.

Assume now that the first two hypotheses of the theorem are satisfied, and, in addition, that $f'(a+)$ exists. The bounds M and N can then be chosen as any two quantities enclosing $f'(a+)$, however tightly, by taking b sufficiently close to a. This guarantees the existence of $f'_+(a)$ and three corollaries follow *mouches à merde*:

I. If $f(a+)$ *exists and is finite, and* $f'(x)$ *exists on some interval* (a, b) *with* $f'(a+)=\infty$, *then* $f'_+(a)=\infty$;

II. If $f(a+)$ *exists and is finite, and* $f'(x)$ *exists on some interval* (a, b) *with* $f'(a+)=-\infty$, *then* $f'_+(a)=-\infty$;

III. If $f(a+)$ *exists and is finite, and* $f'(x)$ *exists on some interval* (a, b) *with* $f'(a+)$ *finite, then* $f'_+(a)=f'(a+)$.

A closely related result appears as a fourth corollary:

IV. If f *is differentiable on an open interval* (a, b) *and* $f'_+(a)$, $f'(a+)$ *both exist (finite valued or otherwise) then they are equal.*

Proof. The existence of $f'_+(a)$ presupposes the existence of a finite-valued $f(a+)$. The hypotheses for one of the three preceding corollaries are then satisfied, and the present conclusion follows immediately.

If $f'(x)$ exists and is finite on some open interval with a as left end point and $f'_+(a)$ exists, Corollary IV covers all possibilities except one: $f'(x)$ may oscillate as $x\to a^+$. Even in this case, $f'(x)$ converges to $f'_+(a)$, provided we take the limit in the mean-value sense. Thus we have the theorem:

If

(i) $f'_+(a)$ *exists (finite valued or otherwise)*;

(ii) $f'(x)$ *exists and is finite on some open interval* (a, b); *then*

$$\lim_{x\to a^+} \frac{1}{(x-a)} \int_a^x f'(t)dt = f'_+ (a).$$

Proof. As before, there is no loss of generality in assuming that f is continuous on the right at $x=a$. Consider, then, any $x\in(a, b)$. With the assumed hypotheses, the integral appearing in the conclusion of theorem is guaranteed to exist, at least in the Denjoy sense. (For those readers unfamiliar with advanced theories of integration we remark that the Denjoy integral (or Denjoy-Perron integral as it is sometimes called) is a generalization of the Riemann integral which is sufficiently broad to reduce the Fundamental Theorem of Calculus to a tautology.) Moreover the Fundamental Theorem of Calculus applies, i.e.,

$$\int_a^x f'(t)dt = f(x) - f(a+).$$

Dividing both sides by $x-a$ and passing to the limit yields the conclusion of the theorem.

The results of this section show that if f is differentiable on some interval having a as left end point, and if we assume that $f'_+(a)$ exists (or, equivalently, that $f(a+)$ and $f'(a+)$ both exist), then the behavior of $f'(x)$ as $x\to a^+$ is governed quantitatively by the value of $f'_+(a)$: if $f'(a+)$ exists, it equals $f'_+(a)$; if it does not exist, $f'(x)$ converges in the mean-value sense to $f'_+(a)$ as $x\to a^+$. Moreover, these results are obtained entirely through rudimentary concepts of analysis: there is no appeal to uniform continuity, bounded variation, continuous differentiability, or Lipschitz conditions.

(For pedagogical purposes, however, one might wish to avoid mention of the Denjoy integral in the proof of the second theorem. For this purpose it is necessary to impose a boundedness condition guaranteeing that f' is improperly Riemann-integrable (a less restrictive assumption than Lebesgue integrability when one treats the highly oscillatory functions for which $f'(a+)$ fails to exist). An extensive discussion of general theories of integration and their relation to the Fundamental Theorem of Calculus is available in [**2**].)

The example of section 2 illustrates the importance of the hypothesis that f be differentiable over an interval. In the next section, we investigate the effects of removing or weakening the other hypothesis, viz, that $f'_+(a)$ exists. The results of the present section are partially recovered, but the mathematical argument is of a somewhat less intuitive character.

4. Results which follow from specifying the behavior of $f'(x)$ as $x\to a^+$. As in section 3, we assume that $f'(x)$ exists and is finite through some interval (a, b). If we specify the behavior of $f'(x)$, but drop the *a priori* assumption that $f'_+(a)$ exists, we are faced with a question more fundamental than the existence and permitted values of $f'_+(a)$, viz., the existence of $f(a+)$. Unless $f(a+)$ exists and is finite, it is pointless to discuss $f'_+(a)$.

If $f'(x)$ becomes unbounded as $x \to a^+$, the existence of $f(a+)$ is not assured, even if f is differentiable infinitely often on (a, b). Consider, for example, the functions

$$(4) \qquad u(x) = \ln x, \quad v(x) = x^{-1}, \quad w(x) = \sin x^{-1} \ (0 < x < 1).$$

As $x \to 0^+$, $u'(x) \to \infty$, $v'(x) \to -\infty$, $w'(x)$ oscillates with infinite amplitude; none of the functions (4) approaches a finite limit as $x \to 0^+$. The issue can always be decided through the Fundamental Theorem of Calculus, which applies on all closed subintervals $[x, B]$ of (a, b). Thus, for B fixed in (a, b), either

$$f(B) - \lim_{x \to a^+} \int_x^B f'(t)dt$$

exists or it does not; if it does exist, it is $f(a^+)$.

On the other hand, if f has a bounded derivative on some interval (a, b), then $f(a+)$ exists and is finite. This result is by no means new, but since it is not too well known, we supply a proof:

By hypothesis, there exists a positive quantity K such that $|f'(x)| < K$ throughout (a, b). Given any preassigned positive quantity δ, choose x and y to be different points of the subinterval $(a, a+\delta/K)$. By the Law of the Mean, there is at least one point z between x and y such that $f(x) - f(y) = (x-y)f'(z)$. Hence,

$$|f(x) - f(y)| = |x - y| \, |f'(z)| < \frac{\delta}{K} \cdot K = \delta,$$

which is the Cauchy criterion for existence of $\lim_{x \to a+} f(x)$.

We obtain a concomitant strengthening of Corollary III to the main theorem of the previous section:

If f is differentiable on some interval (a, b) with $f'(a+)$ finite, then $f'_+(a) = f'(a+)$.

Suppose now that f' is bounded on (a, b) but $f'(a+)$ does not exist, i.e., $f'(x)$ oscillates with finite amplitude as $x \to a^+$. As before, $f(a+)$ exists. The three hypotheses of the main theorem in section 3 are then satisfied, and the bounds M, N are finite. In the proof of that theorem, we established that

$$M < \frac{f(x) - f(a+)}{x - a} < N$$

for each x in (a, b). Consequently, $f'_+(a)$ cannot equal $\pm\infty$, nor can $(x-a)^{-1}[f(x) - f(a+)]$ oscillate with infinite amplitude. Two possibilities remain: either $f'_+(a)$ exists and is finite, or $(x-a)^{-1}[f(x) - f(a+)]$ oscillates with finite amplitude. These are illustrated, respectively, by $x^2 \sin x^{-1}$ and $x \sin (\ln x)$ in the interval $(0, 1)$.

Cases in which $f(a+)$ exists and is finite, and $f'(a+)$ exists but is not finite are covered by Corollaries I and II of the previous section.

The one remaining case—$f'(x)$ undergoes unbounded oscillation as $x \to a^+$—admits of the most possibilities for $f'_+(a)$. First of all, there is no assurance that $f(a+)$ exists. Even if it does, five possibilities remain: $f'_+(a)$ may be finite, $+\infty$, or $-\infty$; $(x-a)^{-1}[f(x)-f(a+)]$ may oscillate with bounded or unbounded amplitude as $x \to a^+$. For illustration, consider the five functions $x^2 \sin x^{-2}$, $x^2 \sin x^{-2} + \sqrt{x}$, $x^2 \sin x^{-2} - \sqrt{x}$, $x \sin x^{-1}$, $\sqrt{x} \sin x^{-1}$.

Those cases for which $f'_+(a)$ exists whereas $f'(x)$ undergoes bounded or unbounded oscillation as $x \to a^+$ were covered in section 3. It was established that $f'(x)$ converges, in the mean value sense, to $f'_+(a)$. The converse is also true, for we have the theorem:

If

(i) $f'(x)$ *exists and is finite on some open interval* (a, b);

(ii) $\lim_{x \to a^+} \left[\frac{1}{x - a} \lim_{y \to a^+} \int_y^x f'(t)dt \right] = A$ *(finite or otherwise), the integral being taken in the Denjoy sense* (If one stipulates that the integral be taken in the Riemann sense, the theorem is true *a fortiori*.); *then* $f'_+(a) = A$.

Proof. With hypothesis (i), the Fundamental Theorem of Calculus holds on each subinterval $[x, y]$ of (a, b). Hypothesis (ii) may therefore be written

$$\lim_{x \to a^+} \frac{f(x) - \lim_{y \to a^+} f(y)}{x - a} = A$$

which has meaning only if $\lim_{y \to a^+} f(y)$, i.e., $f(a+)$, exists. Thus,

$$\lim_{x \to a^+} \frac{f(x) - f(a^+)}{x - a} = A.$$

But the limit on the left side of this equation is $f'_+(a)$.

5. Summary. As we demonstrated in section 2, it is easy to construct examples of functions for which $f'_+(a)$ and $f'(a+)$ are different, provided we permit $f'(x)$ to fail to exist on a set of points which have a as a left-hand limit point. However, if $f'(x)$ exists and is finite on some interval (a, b), the value of each of these quantities strongly influences the other. As direct consequences of this differentiability condition, we have established:

(1) If $f'_+(a)$ and $f'(a+)$ both exist, finite-valued or otherwise, they are equal.

(2) If $f(a+)$ and $f'(a+)$ both exist, so does $f'_+(a)$, which then equals $f'(a+)$. If $f'(a+)$ is finite, the assumption that $f(a+)$ exists is superfluous;

(3) $f'_+(a)$ exists, finite-valued or otherwise, whilst $f'(a+)$ does not, if and only if $f'(x)$ converges to $f'_+(a)$ as $x \to a^+$, not as a limit but in the mean-value sense.

Elementary examples of the various possibilities are given in the table. Missing entries have been proved impossible.

In closing, let us consider briefly those functions which are differentiable on some interval (a, b), with $f(a+)$ existing and finite, but for which $f'(x)$ does not converge—even in the mean-value sense—as $x \to a^+$. For a function of this sort, neither $f'_+(a)$ nor $f'(a+)$ exists. However, it may be possible to find a pair of higher order limit processes which cause $f'(x)$ and $(x-a)^{-1}[f(x)-f(a+)]$ to converge to definite values as $x \to a^+$. From the present results, one might conjecture that the two values are equal, but the question remains open at this time.

References

1. J. M. H. Olmsted, Advanced Calculus, Appleton-Century-Crofts, New York, 1961.
2. J. P. Natanson, Theory of Functions of a Real Variable, Ungar, New York, 1961.

BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics, San Jose State College, San Jose, California 95114.

Challenging mathematical problems with elementary solutions. Volume I: Combinatorial analysis and probability theory. By A. M. Yaglom and I. M. Yaglom. Translated by James McCawley, Jr., Revised and edited by Basil Gordon. Holden-Day, San Francisco, 1964. viii+231 pp. $5.95.

This book is the first of a two-volume translation and adaptation of a well-known Russian problem book entitled *Non-Elementary Problems in an Elementary Exposition.* The authors are twin brothers who have done much in the area of mathematics education in the Soviet Union.

The volume is divided into three sections: Problems, Solutions of Problems, and Answers and Hints. There are 100 problems divided into the following groups:

1. Introductory problems (including the problem of painting a cube)
2. Representations of integers as sums and products
3. Combinatorial problems on the chessboard
4. Geometric problems on combinatorial analysis
5. Problems on binomial coefficients
6. Problems on computing probabilities

7, 8. Experiments with infinitely many outcomes, or with a continuum of possible outcomes.

In addition to the problems themselves, the authors include some definitions and supplementary material. There are six pages of text introducing the section some interval (a, b), with $f(a+)$ existing and finite, but for which $f'(x)$ does not converge—even in the mean-value sense—as $x \to a^+$. For a function of this sort, neither $f'_+(a)$ nor $f'(a+)$ exists. However, it may be possible to find a pair of higher order limit processes which cause $f'(x)$ and $(x-a)^{-1}[f(x)-f(a+)]$ to converge to definite values as $x \to a^+$. From the present results, one might conjecture that the two values are equal, but the question remains open at this time.

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In addition to the problems themselves, the authors include some definitions and supplementary material. There are six pages of text introducing the section on Experiments with a Continuum of Possible Outcomes. In discussing the term "at random" in this section the authors use the classical problem "What is the probability that a chord drawn at random on a circle will be longer than the side of the inscribed equilateral triangle?", with three possible probabilities depending on the application of "at random."

The second part of the volume consists of about 180 pages and contains complete well-developed solutions of the problems with alternate solutions in some cases. The Answers and Hints section gives numerical answers and hints for some of the problems. Thus the student can solve a problem and then compare his answer with the solution provided in the volume.

There are many problems that can be solved as soon as the reader has a basic knowledge of combinatorial analysis. In addition there are included problems of greater difficulty some of which require only more problem-solving background. These are indicated with one, two, or three asterisks. Those with more than one asterisk are often very difficult and may have represented important achievements. In these cases the detailed solutions are valuable in developing further background. In addition to being of interest to problem solvers this volume should prove very useful for mathematics clubs or for high school and undergraduate research projects.

The problems in this volume are nicely grouped so as to develop generalizations; this should be very helpful for research projects. Some examples of problems are given below:

Problem 31a. How many different solutions in positive integers does the equation $x_1+x_2+\cdots+x_m=n$ have?

Problem 44b. What is the greatest number of parts into which a plane can be divided by n circles?

Problem 75a. Two hunters see a fox and shoot at it simultaneously. Assume that each of the hunters averages one hit per three shots. What is the probability that at least one of the hunters will hit the fox?

Problem 90. What is the probability that the first digit of 2^n is a 1?

The second volume on *Problems from Various Branches of Mathematics* will be eagerly awaited.

I. D. RUGGLES, Stanford Research Institute

Introduction to mathematics. By B. E. Meserve and M. A. Sobel. Prentice-Hall, Englewood Cliffs, N.J., 1964. ix+290 pp. $6.25.

The audience for introductory courses in mathematics has grown considerably over the past few years and so has the number of books which have been written to serve as textbooks for such courses. Some of these books are too hard for an introductory course; some are too easy. *Introduction to Mathematics* by Bruce E. Meserve and Max A. Sobel represents a very satisfactory middle ground text. The preface to the text suggests that it was meant for a wide variety of audiences who need or desire some introductory work in mathematics. The depth and scope of the material presented is well taken to fulfill this aim.

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The systematic treatment begins in the second chapter with a study of systems of numeration which is followed by several chapters devoted primarily to number systems and their properties. There are also two chapters dealing with topics in geometry as well as the material on sets, probability, algebra, and logic. The chapters on geometry are especially nice. Most books written for this audience have little beyond an introduction to geometry and yet this is material which can be both interesting and challenging to the elementary student. The authors have covered ideas from intuitive geometry to analytic geometry with some emphasis also given to such topics as non-Euclidean and projective geometry.

The exposition in general is clear and to the point. It is free from rather trite and home-spun examples which seem to plague so many books written for this audience. If the instructor wishes to develop topics in his own style, the exposition will augment rather than get in the way of such treatment. The selection of problems is good as well as abundant. Because the treatment of each topic is quite economical, self study of this volume might be difficult. For the classroom this economical presentation is surely much more desirable than those presentations which are so long that the main ideas are obscured.

There are those who might object to the fact that the formal treatment of sets is delayed until the fourth chapter but this unit is sufficiently independent that it may be presented earlier if desired.

This reviewer, who has used the volume as a text, feels that the book is more than adequate when used for an introductory course at essentially lower division level and for general education purposes.

R. B. BRIAN, San Jose State College

Calculus with analytic geometry, 3rd Ed. By R. E. Johnson and F. L. Kiokemeister (with Exercises by M. S. Klamkin). Allyn and Bacon, Boston, 1964. xii+798 pp. $14.60.

The third edition of this best seller shows rather extensive revision over the first and second editions. The changes are as follows: (1) the trend toward increased rigor and greater precision in statements, nomenclature, and notation is continued; (2) more topics from what used to be called advanced calculus are included; (3) the problem lists have been completely revised.

The breadth of coverage is indicated by this quotation from the authors' preface: "On the one hand, the notation of set theory has been incorporated, and on the other hand, the text has been extended to include Green's theorem. . . . Lack of space forced us to stop short of Stokes's theorem, but the necessary groundwork is there for the enterprising teacher who wishes to forge ahead on his own."

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Users of previous editions will find that rearrangements of material have frequently improved the order. The first chapter is greatly improved. Vectors are used to great advantage throughout the latter half. The notation in Chapter 19: Line and Surface Integrals, is a bit burdensome at times, such as the description of the Jacobian matrix of a mapping F whose domain and range are in R_2, described as follows:

$$F(P) = (F^1(P), F^2(P)) \quad \text{and} \quad J_F(P) = D_1F^1(P)D_2F^2(P) - D_1F^2(P)D_2F^1(P).$$

Problem lists are separated into Parts I and II, the latter list being composed of more challenging ones. Honor students will find this book challenging and exciting. Most students will find it more difficult than the earlier editions.

R. C. MEACHAM, Florida Presbyterian College

A first course in calculus. By Serge Lange. Addison-Wesley, Reading, Mass., 1964. xii+258 pp. $6.75.

At a time when most authors of calculus texts are bringing out texts which are encyclopedic in size and content, along comes this beautiful counterexample of clean expository simplicity. This volume is the first of two and it covers the calculus of one variable. (Volume two is slightly shorter and treats calculus of several variables and linear algebra.) The author has attempted to present the essentials of calculus in a form which the student will find readable, intuitive, and yet rigorous. By not insisting on telling more than is absolutely necessary he achieves a remarkably uncluttered book.

It is the author's thesis that many potential mathematics students are capable of understanding the beauty and power of mathematics at an early stage—provided that he is not frightened away by misplaced rigor. He feels that most of the ideas in this volume can be adequately presented in secondary school, even!

Perhaps the greatest time-and-space saver employed by the author is the avoidance of epsilon-delta type proofs in the body of the text. For completeness he treats the fundamental limit theorems in complete rigor in the appendix, but avoids this type proof throughout the rest of the book. Nevertheless the phraseology of all proofs is such that one who wishes to use inequalities throughout could do so.

Topics covered include graphs, curves, derivative, sine and cosine, mean value theorem, inverse functions, exponents and logarithms, integration, Taylor's formula, series. This wealth of topics is treated rigorously and yet the student is led to feel that he is a party to the proof. He is led into guessing the theorem frequently, and then proving that the guess is correct. Answers are given to all exercises.

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R. C. MEACHAM, Florida Presbyterian College

A second course in calculus. By Serge Lange. Addison-Wesley, Reading, Mass., 1964. xii+242 pp. $7.50.

This volume is the companion to the author's First Course in Calculus. It is intended for students who have had two semesters of calculus of one variable. The first half of the book treats calculus of several variables and the second half linear algebra. Chapter I is on vectors, which are then used throughout the remainder.

As in the first volume, the author manages to present the proofs in remarkably clear, uncluttered form, relegating to the appendices some of the techniques and proofs which, if employed throughout, would have considerably lengthened the text. The instructor will perhaps wish to include these techniques in many of the proofs, however.

Topics covered include differentiation of vectors, functions of several variables, chain rule and gradient, potential functions and line integrals, Taylor's formula, maximum and minimum (including Lagrangian multipliers), multiple integrals, vector spaces, linear equations and bases, linear mappings, linear maps and matrices, applications to functions of several variables, determinants, and complex numbers. The appendix includes a brief presentation of mathematical induction, limit theorems for normed vector spaces, and analytic definitions for sine, cosine, and angle.

The two halves of the book, after Chapter I and excepting the applications in Chapter XIII to functions of several variables, could be taught in either order.

R. C. MEACHAM, Florida Presbyterian College

The Mathematics of Matrices. By Philip J. Davis. Blaisdell, New York, 1965. xiii+348 pp. $7.50.

This very well done text has been written for students in the last year of high school or first year college freshmen. As the subtitle implies it is an elementary treatise on matrix theory and linear algebra. In this writer's opinion, the book would also be a good choice for a general education course in mathematics at the freshman or sophomore level.

The book abounds with excellent practical examples and many references to the history of determinants, matrices, and quaternions as well as other topics in mathematics. The problem sets are numerous and well chosen; answers are not provided in the book.

Obviously at this level examples of good applications of the material cannot be over done. The only significant example which is missing from the text is the application of matrices to problems in genetics which affords an excellent elementary example of how powers of matrices can be used in a most practical way.

Many topics are covered in a full and careful way so that an instructor may easily pick and chose among the ten chapters in order to build a course which will be most appropriate to the abilities and needs of a particular class of students.

D. G. DUNCAN, Sonoma State College

The theory of numbers. By Neal H. McCoy. Macmillan, New York, 1965. ix+150 pp. $4.95.

It is a commonplace that the very framework within which mathematical research is carried on is, in our age, undergoing a rapid and accelerating evolution. One hundred years ago, the mathematical world was only just getting used to the notion of a general function or even of a general analytic function, as opposed to specific (e.g. elliptic) functions; less than seventy years ago, a leading group theoretician (Klein) is reported to have rejected the notion of an *abstract* group; less than sixty years ago an eminent algebraist (Schur) was heard to comment with some asperity on the "needless generality" of Steinitz' theory of fields;—yet nowadays there are those who wish to supersede even the remarkably comprehensive algebraic-topological framework established by the renowned N. Bourbaki, by an even more general and more abstract system.

At a time when the reverberations of this development can be heard in the elementary schools (and in "Pogo"), it is natural that the question to what extent undergraduate teaching should adapt itself to the latest (or perhaps, to the prepenultimate) approach, is a matter of much discussion. In some fields, e.g., in Algebra, a fairly rapid evolution of undergraduate texts is mandatory. In other disciplines, conspicuously so in elementary Number Theory, the decision to accept modern terms is largely a matter of taste. There is in fact no need to be unduly dogmatic in solving this problem in relation to an individual course so long as the student's total curriculum is well integrated and introduces him both to the depth and beauty of classical methods and to the coherence and broad outlook of the modern approach.

The author of the book under review has chosen to present the elements of the Theory of Numbers in traditional terms. He has produced a very readable text which is well summarized by the chapter headings—1. Divisors and prime numbers, 2. Fundamental properties of congruence, 3. Polynomial congruences and primitive roots, 4. Quadratic residues, 5. Continued fractions. In the reviewer's opinion, the book can be highly recommended as a text for a first course in Number Theory so long as it is balanced by an early course in a more abstract spirit, e.g., an elementary course in Algebra in the spirit of Professor McCoy's own attractive introduction to the Theory of Rings.

ABRAHAM ROBINSON, U.C.L.A.

Acknowledgement of Priority: M. S. Klamkin, the author of "An Extension of the Butterfly Problem," this MAGAZINE, Sept., 1965, has just noted that the result is not new. R. A. Johnson in his *Advanced Euclidean Geometry*, Dover, New York, 1960, p. 78, gives a relatively simple geometrical proof which he attributes to Mackay, Proc. Edinburgh. Math. Soc. III, 1884, p. 38.

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted. Solutions should be submitted on separate, signed sheets. Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles, California 90029.

PROBLEMS

614. *Proposed by Charles W. Trigg, San Diego, California.*

In the cryptarithm

$$V\ E\ X\ I\ N\ G = M\ A\ T\ H\ ,$$

the X doubles as a multiplication sign and each other letter uniquely represents a positive digit. $M\ A\ T\ H$ is a permutation of consecutive digits. Find the two numerical solutions.

615. *Proposed by Joseph Hammer, University of Sydney, Australia.*

Prove that in a three-dimensional convex surface whose volume is greater than the surface area numerically, infinitely many plane cross-sections can be found of which each area is greater than its perimeter.

616. *Proposed by Rosemary Griffith, Technical Operations Research, Burlington, Massachusetts.*

(a) The factor $6/(N^3-N)$ appears in the Spearman rank correlation coefficient, where N is the sample size. Show that this factor can be reduced to the form $1/M$, M an integer.

(b) In general, determine the restrictions on n (an integer) and m (a positive integer) so that the expression (n^m-n) is divisible by $m!$.

617. *Proposed by Norman Schaumberger and Erwin Just, Bronx Community College, New York.*

A regular polygon of $2n$ sides has a unit radius and vertices $A_i (i=1, 2, \cdots, 2n)$. If a_k is the length of the line segment $A_{k+1}A_1$, prove that

$$\prod_{i=1}^{n-1} a_i^2/(4 - a_i^2) = 1.$$

618. *Proposed by Albert Wilansky, Lehigh University.*

Let A and B be n by n idempotent $(A^2=A)$ matrices of real numbers such that $(A-B)^2=0$. Then either A and B have the same range or they have the same nullspace. Prove that this is true for $n=2$, 3 and false for $n\geqq 4$.

619. *Proposed by Alan Sutcliffe, Congleton, Cheshire, England.*

In a triangle ABC, right-angled at C, the bisectors of angles A and B meet BC and AC at D and E, respectively. If $CD=9$ and $CE=8$, what are the lengths of the sides of the triangle?

620. *Proposed by Daniel B. Lloyd, District of Columbia Teachers College.*

(a) The town clock in Zurich was started one evening at 6 o'clock in a manner to cause the natives to believe it to be bewitched. The hour and minute hands had been accidentally interchanged, causing the hour hand to rotate twelve times as fast as the minute hand. How soon thereafter would the hands tell exactly the correct time?

(b) The clock repairman, duly embarrassed, labored feverishly all night to correct his error. However, in the confusion, and with poor light, the clock pinions became interchanged. When the clock was started again at 6 o'clock the next morning, the hands appeared correct on the face, but the hour hand again started rotating twelve times as fast as the minute hand. However, when the town officials came later in the morning to inspect the work, the clock showed the correct time. What time was it then?

Erratum: Problem 577 (Jan, 1965) was proposed by *M. S. Klamkin, SUNY at Buffalo and L. A. Shepp, Bell Telephone Laboratories.*

SOLUTIONS

A Trigonometric Inequality

587. [May, 1965] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Prove the following inequality

$$\left(\frac{\theta+\sin\theta}{\pi}\right)^2+\cos^4\frac{1}{2}\theta<1, \qquad (-\pi<\theta<+\pi).$$

Solution by Samuel Wolf, Linthicum Heights, Maryland.

$$\left(\frac{\theta+\sin\theta}{\pi}\right)^2+\cos^4\frac{\theta}{2}=\left(\frac{\theta+\sin\theta}{\pi}\right)^2+\left(\frac{1+\cos\theta}{2}\right)^2=F$$

Differentiating, and setting to zero:

$$\frac{2}{\pi^2}(\theta+\sin\theta)(1+\cos\theta)=\frac{1}{2}(1+\cos\theta)(\sin\theta)$$

$$\frac{4}{\pi^2}(\theta+\sin\theta)=\sin\theta \qquad [\cos\theta\neq-1]$$

$$\frac{4}{\pi^2}\theta + \sin\theta\left(\frac{4}{\pi^2} - 1\right) = 0. \qquad \theta = 0 \text{ is a solution.}$$

$$\frac{\sin\theta}{\theta} = \frac{4}{\pi^2 - 4} = \frac{4}{9.8696 - 4} = \frac{4}{5.8696}$$

$$\frac{\sin\theta}{\theta} = .6815$$

$$\theta = \pm 1.46 \text{ (Jahnke and Emde, appendix p. 33)}$$

$$F_0 = 1; \qquad F_{\pm 1.46} = \left(\frac{1.46 + .99}{\pi}\right)^2 + \left(\frac{1 + .11}{2}\right)^2 = .92.$$

Taking the second derivative:

$$G = \frac{2}{\pi^2}\left[(1 + \cos\theta)^2 + (\theta + \sin\theta)(-\sin\theta)\right] - \frac{1}{2}\left[-\sin^2\theta + (1 + \cos\theta)\cos\theta\right].$$

For $\theta = 0$, $G < 0$, so $\theta = 0$ is a maximum.
For $\theta = \pm 1.46$, $G > 0$, and $\theta = \pm 1.46$ are minimums.
Thus $F \leqq 1$.
(Note: The "=" sign is necessary.)

Also solved by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; C. B. A. Peck, State College, Pennsylvania; Simeon Reich, Haifa, Israel; Sidney Spital, California State Polytechnic College; K. L. Yocom, South Dakota State University; and the proposer.

Raymond E. Whitney, Lock Haven State College, Pennsylvania, pointed out the necessity of including the equals sign along with the inequality.

What is Best?

593. [September, 1965] *Proposed by J. A. H. Hunter, Toronto, Canada.*

$$\begin{array}{rrrr} & S & E & E \\ & & M & M \\ & S & E & E \\ & & M & M \\ & & M & M \\ \hline B & E & S & T \end{array}$$

Each letter here stands for a particular and different digit. This alphametic expresses a simple truth, for there is nothing at all odd about our $M\ M$. So what is $B\ E\ S\ T$?

I. *Solution by Stephen Hoffman, Trinity College, Connecticut.*

We have $S \neq 0$, $B \neq 0$, and $M \neq 0$. Further,

$$\begin{aligned} 2E + 3M &= 10i + T, & 1 \leqq i \leqq 4; \\ 2E + 3M + i &= 10j + S, & 0 \leqq j \leqq 4; \\ 2S + j &= 10B + E. \end{aligned}$$

The third equation implies $S \geqq 3$. Also, $10i + T = 10j + 3$.

If $j=0$, then $S \geqq 11$. If $j=i+2$, then $T \geqq 16$. If $j=i+3$, then $T \geqq 26$. If $j=i+1$, then $i+T=10+S$ with $1 \leqq i \leqq 3$; $i=1$ implies $S=0$, $i=2$ implies $2S+j=4$, and $i=3$ implies $2S+j=5$ or 7. Thus $i=j$.

For $i=1$, we have $S \geqq 5$ and, of these values, only $S=7$, $T=6$, $B=1$, $E=5$, and $M=2$ yield a solution. For $i=2$, we have $S \geqq 4$ and no values of S yield a solution. For $i=3$, we have $S \geqq 4$ and no values of S yield a solution. For $i=4$, we have $S \geqq 4$ and of these values only $S=7$, $T=3$, $B=1$, $E=8$, and $M=9$ yield a solution.

There are thus two solutions to the cryptarithm:

$$\begin{array}{r} 755 \\ 22 \\ 755 \\ 22 \\ 22 \\ \hline 1576 \end{array} \qquad\qquad \begin{array}{r} 788 \\ 99 \\ 788 \\ 99 \\ 99 \\ \hline 1873 \end{array}$$

and in the second, there is indeed something odd about $M\ M$. Therefore 1576 is $B\ E\ S\ T$.

II. *Solution by Dermott A. Breault, Sylvania Electronic Systems, Waltham, Massachusetts.*

The given sum is equivalent to finding solutions of the following equation in integer values from 0 to 9:

$$T = 33M + 190S - 78E - 1000B.$$

A FORTRAN program was written to examine the 10,000 possible configurations of (M, S, E, B) and print out those for which T had an acceptable value also. This procedure resulted in the list below:

	T	M	S	E	B
1.	9	5	0	2	0
2.	0	4	3	9	0
3.	4	4	5	1	1
4.	5	3	6	3	1
5.	6	2	7	5	1
6.	7	1	8	7	1
7.	3	9	7	8	1
8.	8	0	9	9	1
9.	7	9	9	0	2

Repeated digits rule out all but numbers 5. and 7. above, and the additional requirement that M be even makes 5. a unique solution, hence $B\ E\ S\ T=1576$.

(The program took ten minutes to write and five seconds to run on a CDC 3200.)

Also solved by Merrill Barneby, Wisconsin State University at LaCrosse; Murray Berg, Standard Oil Company, San Francisco, California; Barbara Boettger, Canton, Georgia; Maxey Brooke, Sweeny, Texas; James Callan, University of Oklahoma; Peg Duff, Vassar College; Dewey Duncan, Los Angeles, California; Frederick K. Fidler, Adelphi, Maryland; Herta T. Freitag, Hollins College, Virginia; Philip Fung, Cleveland State University, Ohio; H. M. Gehman, SUNY at Buffalo, New York; Sidney Kravitz, Dover, New Jersey; James W. Lea, Jr., University of Tennessee at Martin; Sam Lesseig, State Teachers College, Kirksville, Missouri; Marilee Logsdon, Chiddix-Junior High School, Normal, Illinois, John W. Milsom, Slippery Rock State College, Pennsylvania; Richard Riggs, Jersey City State College; John W. Schestedt, Stigler, Oklahoma; C. W. Trigg, San Diego, California; John Waddington, Levack, Ont., Canada; Howard L. Walton, Yorktown High School, Arlington, Virginia; Jane Weinberg, Vassar College; Hazel S. Wilson, St. Petersburg, Florida; Dale Woods, Northeast Missouri State Teachers College; K. L. Yocom, South Dakota State University; and the proposer. One incorrect solution was received.

Another Triangle Inequality

594. [September, 1965] *Proposed by Leon Bankoff, Los Angeles, California.*

If R is the circumradius, r the inradius, and AD, BE, CF the altitudes of triangle ABC, show that

$$AD + BE + CF \leqq 2R + 5r.$$

Solution by Dale Woods, Northeast Missouri State Teachers College.

Let H be the orthocenter, O be the circumcenter, A', B', C' be the feet of the perpendiculars from O to BC, CA and AB respectively. It is quite well known that $AH=2OA'$; $BH=2OB'$; $CH=2OC'$ and that $OA'+OB'+OC'=R+r$ therefore

$$AD + BE + CF = AH + HD + BH + HE + CH + HF = 2R + 2r + HD + HE + HF \leqq 2R + 5r \quad \text{since} \quad HD + HE + HF \leqq 3r.$$

Also solved by W. J. Blundon, Memorial University of Newfoundland; Dewey Duncan, Los Angeles, California; Stephen Hoffman, Trinity College, Connecticut; G. L. N. Rao, J. C. College, Jamshedpur, India; Paul D. Thomas, US Naval Oceanographic Office, Suitland, Maryland; and the proposer.

A Repeating Decimal

595. [September, 1965] *Proposed by Douglas Lind, University of Virginia.*

Form the decimal number N in the following manner: the nth digit of N is the sum of the units digit of n, the tens digit of $n+1$, the hundreds digit of $n+2$, etc. If this sum is greater than ten, the excess is carried over in the usual way. Prove that N is a repeating decimal and find its fractional equivalent.

Solution by Dewey Duncan, Los Angeles, California

As an infinite series, the number must appear as

$$N = (1/10) + 2(1/10)^2 + 3(1/10)^3 + 4(1/10)^4 + \cdots,$$

evaluated finitely thus:

$$(1/10)N = (1/10)^2 + 2(1/10)^3 + 3(1/10)^4 + \cdots$$

whence

$$N - (1/10)N = (1/10) + (1/10)^2 + (1/10)^3 + \cdots = 1/9.$$

Finally $N=10/81$, or in decimal form, .123456790123456790 · · · of period 1 2 3 4 5 6 7 9 0.

Also solved by E. S. Langford, U. S. Naval Postgraduate School and the proposer.

An Impossible Situation

596. [September, 1965] *Proposed by William K. Viertel, State University Agricultural and Technical College, Canton, New York.*

Prove that there is no area bounded by a curve of the family $y=x^n$, $(n>0)$, the x-axis, and the line $x=a$, for which:

(a) the abscissa of the centroid, $\bar{x}$, is the same as the radius of gyration R_y with respect to the y-axis; or

(b) the abscissa of the centroid, $\bar{x}_a$, is the same as the radius of gyration R_a with respect to the line $x=a$; or

(c) the ordinate of the centroid, $\bar{y}$, is the same as the radius of gyration R_x with respect to the x-axis.

Solution by K. L. Yocom, South Dakota State University.

The usual formulas from the calculus when applied to the family $y=x^n$ yield

$$\bar{x} = \frac{a(n+1)}{n+2}, \qquad \bar{y} = \frac{a^n(n+1)}{2(2n+1)} \qquad \bar{x}_a = \frac{a}{n+2},$$

$$R_x^2 = \frac{a^{2n}(n+1)}{3(3n+1)}, \qquad R_y^2 = \frac{a^2(n+1)}{n+3}, \qquad R_a^2 = \frac{2a^2}{(n+2)(n+3)}$$

Applying the conditions of the problem leads to

(a) $\bar{x}^2=R_y^2$ implies $3=4$.
(b) $\bar{y}^2=R_x^2$ implies $7n^2+4n+1=0$ for which n is complex.
(c) $\bar{x}_a^2=R_a^2$ implies $n=-1$.

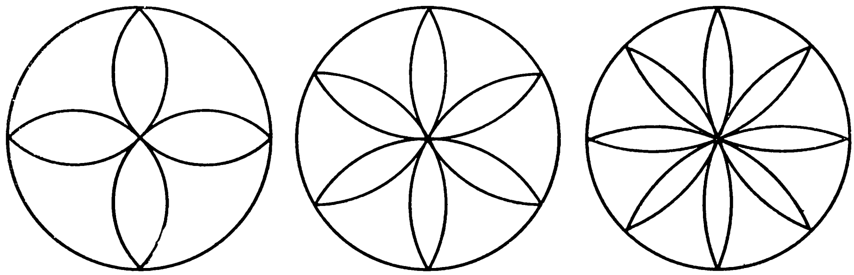
Thus the required areas do not exist.

Also solved by Joseph G. Bohac, St. Louis, Missouri; Phillip Fung, Cleveland State University, Ohio; John Kieffer, University of Missouri at Rolla; E. S. Langford, U. S. Naval Postgraduate School; John Wessner, Melbourne, Florida; and the proposer.

A Covering Theorem

597. [September, 1965] *Proposed by Alan Sutcliffe, Congleton, Cheshire, England.*

It is possible to draw regular rosettes inside a circle with any even number of leaves greater than 2. The first 3 such rosettes are shown here. What is the limit of the proportion of the area of the circle that the leaves cover as their number increases without bound?



Solution by C. Stanley Ogilvy, Hamilton College, New York.

Let the radius of the given circle be a. One assumes from the sketches that the rosettes are composed of circular arcs. Then each leaf of a rosette of n leaves has twice the area of a lune cut off by one side of regular inscribed n-gon of side a. The area of such a lune is

$$\tfrac{1}{2}r^2\left(\frac{2\pi}{n} - \sin\frac{2\pi}{n}\right),$$

where

$$r = \frac{a}{2\sin(\pi/n)}$$

Hence we must evaluate

$$\lim_{n\to\infty} n\left(\frac{a}{2\sin(\pi/n)}\right)^2\left(\frac{2\pi}{n} - \sin\frac{2\pi}{n}\right).$$

This is equivalent to

$$\lim_{n\to 0} \frac{\pi a^2}{2}\left[\frac{1-(\sin x)/x}{\sin^2(x/2)}\right].$$

Three applications of l'Hospital's Theorem yield the limit $\pi a^2/3$, or 1/3 the area of the circle.

It is immaterial whether n is even or odd.

Also solved by Martin J. Brown and Clifford J. Swanger (jointly) University of Kentucky; Dewey Duncan, Los Angeles, California; Mrs. A. C. Garstang, Boulder, Colorado; John Kieffer, University of Missouri at Rolla; J. D. E. Konhauser, University of Minnesota; E. S. Langford, U. S. Naval Postgraduate School; Lieselotte Miller, Georgia Institute of Technology; Garth Peterson, South Dakota School of Mines and Technology; Stanley Rabinowitz, Far Rockaway, New York; K. L. Yocum, South Dakota State University; and the proposer. Four incorrect solutions were received.

The Soda Straw

598. [September, 1965] *Proposed by H. Tracy Hall, Brigham Young University.*

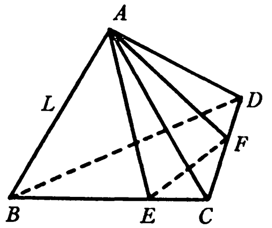
Given a flexible, thin-walled cylinder, such as a soda straw, of diameter D. What is the edge length L of the largest regular tetrahedron that can be pushed through the straw?

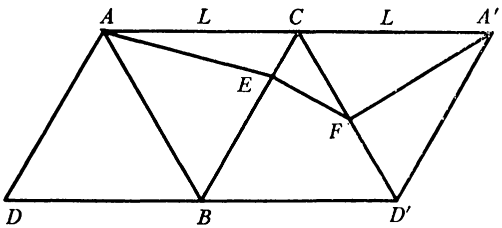
Solution by Charles W. Trigg, San Diego, California.

In a parallelogram consisting of a strip of four equilateral triangles, lines drawn parallel to a long side have a constant length, $2L$. When the strip is folded into a regular tetrahedron, it follows that the sections of the tetrahedron made by planes perpendicular to the join of the midpoints of two opposite edges have a constant perimeter, $2L$.

Consequently, when its bimedian coincides with the axis of the cylinder, the tetrahedron may be pushed through a flexible thin-walled cylinder with a circumference, $\pi D = 2L$. That is, $L = \pi D/2$. In practice, it would be helpful to have the end of the cylinder flared out slightly in order to get the job under way.

If the tetrahedron has a different attitude to the axis of the cylinder, some plane perpendicular to the axis will pass through a vertex and cut two edges not issuing from that vertex. As can be seen from the developed surface in the figure, the perimeter of a typical section $AEFA'$ is greater than $2L$. Consequently, the tetrahedron cannot pass through the cylinder in this attitude. It follows that the largest tetrahedron that can pass through the cylinder is the one with edge $L = \pi D/2$.





Also solved by the proposer. One incorrect solution was received.

Linearly Dependent Vectors

599. [September, 1965] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

If a, b, and c are any three vectors in 3-space, then show that the vectors

$$a\times(b\times c),\ b\times(c\times a),\ c\times(a\times b)$$

are linearly dependent.

Solution by Carl G. Wagner, Duke University.

By a well-known theorem of the vector calculus (see page 90 of Nickerson, Steenrod, and Spencer's *Advanced Calculus* for a proof based on axioms for a vector product):

$$A \times (B \times C) = (A\cdot C)B - (A\cdot B)C.$$

Writing out the other vector products,

$$B \times (C \times A) = (B\cdot A)C - (B\cdot C)A = (A\cdot B)C - (B\cdot C)A$$
$$C \times (A \times B) = (C\cdot B)A - (C\cdot A)B = (B\cdot C)A - (A\cdot C)B.$$

Hence,

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0$$

(This is known as the Jacobi Identity.)

Also solved by Joseph B. Bohac, St. Louis, Missouri; Dermott A. Breault, Sylvania Applied Research Laboratory, Waltham, Massachusetts; Dewey C. Duncan, Los Angeles, California; Philip Fung, Cleveland State University, Ohio; Mrs. A. C. Garstang, Boulder, Colorado; Carl Harris, Western Electric Company, Princeton, New Jersey; Stephen Hoffman, Trinity College, Connecticut; John E. Homer, Jr., St. Procopius College, Illinois; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Mich.; John Kieffer, University of Missouri at Rolla; E. S. Langford, U. S. Naval Postgraduate School; Lieselotte Miller, Georgia Institute of Technology; Stanley Rabinowitz, Far Rockaway, New York; Kenneth A. Ribet, Brown University; Richard Riggs, Jersey City State College; Howard L. Walton, Yorktown High School, Arlington, Virginia; K. L. Yocum, South Dakota State University; and the proposer.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 378. Find the average area of all triangles which can be inscribed in a given triangle. (It is assumed that the vertices are uniformly distributed over the sides of the given triangle.)

[*Submitted by Murray S. Klamkin and W. J. Miller*]

Q 379. Two unequal regular hexagons $ABCDEF$ and $CGHJKL$ (vertices names clockwise) touch each other at C and are so situated that F, C and J are collinear. Show that:

(a) the circumcircle of BCG bisects FJ, and

(b) triangle BOG is equilateral.

[*Submitted by Leon Bankoff*]

Q 380. Exhibit two irrational numbers A and B such that A^B is a rational number.

[*Submitted by Charles Ziegenfus*]

Q 381. Find the area of a quadrilateral inscribed in a circle and circumscribed about a circle if three of the sides, a, b and c are given.

[*Submitted by Julius G. Baron*]

Q 382. A solid of revolution has the equation, in cylindrical coordinates, $r = -a/z$ where the z-axis points vertically upward. Discuss the motion of a small frictionless object initially set moving at an arbitrary point and with arbitrary velocity over the surface of the solid, as viewed in a plane perpendicular to the z-axis. Assume normal sea-level gravity g.

[*Submitted by Harry W. Hickey*]

(Answers to Quickies on pages 111-112)

$$x \log y = -y \log x, \qquad x, y > 0.$$

An analogous argument to that used above will establish the result for C_2^*, the only difference being as follows. The Implicit Function Theorem is used to deduce that the intersections of a line x = constant with the curve C_2^* are isolated and hence countable.

Finally if $(x, y) \in S - C_1$ then either $x = y$ or (x, y) is on one of the curves C_2, C_3 or C_4. Thus $x = y$ or x and y must satisfy one of the equations (4.2), (4.3) and (4.4). Thus x is rational if and only if y is rational.

References

1. E. J. Moulton, The real function defined by $x^y = y^x$, Amer. Math. Monthly, 23 (1916) 233–237.

2. B.Sz.-Nagy, Introduction to real functions and orthogonal expansions, Oxford University Press, New York, 1965.

3. J. M. H. Olmsted, Advanced calculus, Appleton-Century, New York, 1961.

ANSWERS

A 378. (1) Analytic solution.

$$\overline{A} = \frac{1}{abc} \int_0^a \int_0^b \int_0^c \{A - \tfrac{1}{2}[z(b - y) \sin A + x(c - z) \sin B + y(a - x) \sin C]\} dxdydz,$$

$\overline{A} = A/4$ where A is the area of the given triangle.

(2) Geometric solution. If a series of triangles have a common base and their vertices be in a given finite straight line which is wholly on the same side of the base, the average of all triangles thus formed is that whose vertex is at the middle of the line segment; since for every triangle which exceeds this, there is obviously another just as much less than it. Consequently the mean-inscribed triangle is the one joining the midpoints of the sides, and $\overline{A} = A/4$.

A 379. (a) Let the circumcircle of BCG cut FJ at O. Then since angle $OCG = 120°$ we have $OC = BC - CG$. So

$$OJ = BC - CG + 2CG = BC + CG = FJ/2.$$

(b) The chords BO, OG and GB are equal because the angles OCB, GBO and BOG are equal.

A 380. Consider $(\sqrt{3})^{\sqrt{2}}$. If this representation is rational, we are finished. If not, then the above representation is irrational. Then consider

$$[(\sqrt{3})^{\sqrt{2}}]^{\sqrt{2}} = 3.$$

A 381. Let d be the unknown side. In a circumscribed quadrilateral for any two opposite sides (e.g., for a and c) we have

$$a = s - c$$

where

$$s = \tfrac{1}{2}(a + b + c + d).$$

Thus

$$\tfrac{1}{2}(a + b + c + d) - d = b$$

so

$$d = a + c - b.$$

The area of an inscribed quadrilateral is known to be

$$T = \sqrt{\{(s - a)(s - b)(s - c)(s - d)\}}.$$

Substituting, we have

$$T = \sqrt{abcd}$$

where $d = a + c - b$.

A 382. As viewed in the plane, the object will be accelerated toward the center (i.e., the z-axis) with acceleration

$$g \frac{dz}{dr}$$

which is

$$ag/r^2.$$

Hence it will exhibit an inverse-square planetary orbit: ellipse, hyperbola or parabola.

(Quickies on page 134)

THE RELATION OF $f'_+(a)$ TO $f'(a+)$

W. E. LANGLOIS, IBM Research Laboratory, San Jose, California and
L. I. HOLDER, Gettysburg College

1. Introduction. Advanced Calculus textbooks are usually careful to point out the distinction between the *derivative on the right*, defined as

$$(1) \qquad \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a},$$

PROJECTIVE GEOMETRY

H. S. MacDonald Coxeter, *University of Toronto*

Intended for a course at the undergraduate level, this book presents a synthetic treatment of general projective geometry, stressing relations of incidence and projective transformations.

1964. 162 pp. $5.50

THE MATHEMATICS OF MATRICES:
A First Book of Matrix Theory and Linear Algebra

Philip J. Davis, *Brown University*

In this introductory text on matrices, emphasis has been placed on four aspects of matrix theory arranged in a natural association: the notation and terminology, formal algebra, interpretations of matrices, application of matrices. Suitable for college and junior college students, this text is recommended by the School Mathematics Study Group. *1965. 348 pp. $5.20*

Teacher's Manual available.

NUMBERS AND ARITHMETIC

John N. Fujii, *Merritt College*

This basic classroom text on arithmetic, developed for a pre-algebra review course, is well suited for one semester or one or two quarter courses. Proceeding for specific to general patterns of computations and ideas, observations and questions are used freely to motivate the topics. The text is divided into two parts: Natural Numbers and the Integers, and Rational Numbers and Measurement. Teacher's Manual available. *1965. 559 pp. $8.50*

COLLEGE ALGEBRA

Merlin M. Ohmer, *The University of Southwestern Louisiana, and* Clayton V. Aucoin, *Clemson University*

This text is intended as a high-level review of the standard, two-year high school algebra course. It offers a solid foundation in those topics essential to a systematic study of analytic geometry and calculus. *1966. In press.*

MATHEMATICS: THE STUDY OF AXIOM SYSTEMS

George E. Witter, *Western Washington State College*

Fundamental concepts of mathematical thought are covered in this text, providing a sound understanding of the nature of mathematics for the mature student who does not intend to specialize in the field. The scope of the text is broad, ideally suited for undergraduate courses.

1964. 319 pp. $6.50

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